DYNAMICS OF SOME CLASSES OF LOTKA-VOLTERRA STOCHASTIC OPERATORS ON LOW DIMENSIONAL SIMPLEX

BY

AZIZI BIN ROSLI

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Kulliyyah of Science
International Islamic University Malaysia

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ABSTRACT

Lotka-Volterra (LV) operator has been the subject of study in dynamical systems, notably on the asymptotic behaviour of its trajectory. In this thesis, we introduce a general class of LV operator defined on a simplex. This class of LV operator encompasses most of the previously studied LV operators. Our aim is to study the dynamics of operators derived from it for lower dimensional simplex. Firstly, we provide conditions, under which the operator is a bijection when restricted to 2-dimensional simplex (2-simplex). We showed that under such conditions the operator is a homeomorphism, i.e., the determinant of its Jacobian is non-zero. Previously, it was shown that any LV operator which satisfies f-monotonicity condition is bijective. Here, we disprove its converse by giving an example of a bijective LV operator which is not f-monotone. In the second part, we consider and study the limiting behaviour of a class of LV operator defined on the 2-simplex. We find that its interior fixed point is unique. Then, by constructing a Lyapunov function we show that the limiting set lies in the boundary. We also provide a description of the path taken by its trajectory as the number of iterations tends to infinity. We estimate the time the trajectory spent within a neighbourhood of a vertex, and we find that the trajectory does not remain in any of the neighbourhoods. Thereupon, we proceed to show that the operator has the property of being non-ergodic. Then, a special case of the operator above is considered in the next part, in which we consider a convex combination of this non-ergodic higher order LV operator and a regular quadratic LV operator. We find that fixed points exist at the edge of the 2-simplex when its respective parameter is above a critical value. Furthermore, they are saddle-nodes when parameter exceeds the critical value by certain amount or repelling if otherwise. Then, by imposing a condition to its parameter we construct a Lyapunov function for the operator. We show that, under such condition; the LV operator derived from such combination has the property of being regular, and its trajectory converges to one of the vertices for any initial point other than the edge fixed points and interior fixed point. A full description of the limit set of such a trajectory is also obtained. In the final part, we consider a class of LV operator defined on the 3-dimensional simplex (3-simplex). Besides having a unique interior fixed point, we also show the existence of uncountable fixed points on two of its edges. We construct several Lyapunov functions, by which we find that its trajectory converges to the edges. Then, we prove the existence of a subset of 3-simplex with positive measure such that the trajectory of the operator does not converge for any initial point taken from the set, and the eventual path taken by such trajectory is described. Using similar technique as in the case of 2-simplex, we show that the Cesaro mean of its trajectory diverges, i.e., the operator is non-ergodic. At the end, the dynamics on its 2-dimensional faces (2-faces) are studied. The operator is found to be regular when restricted on its 2-faces, and the limiting set is estimated.

ملخص البحث

مشغل لوتكا–فولتر (LV) هو موضوع لدراسة الأنظمة الديناميكية، ولا سيما على السلوك المقارب لمسار المشغل ثنائي التطابق. في هذه الأطروحة، نقدم فئة عامة من مشغل LV محدد على بسيطة. تشمل هذه الفئة من مشغل LV معظم مشغلي LV التي تمت دراستها سابقًا. هدفنا هو دراسة ديناميكيات المشغلين المشتقة منها للبسيطة ذات الأبعاد المنخفضة. أولًا، نوفر شروطًا، بموجبها يكون المشغل ثنائي التطابق عند تقييده إلى بسيطة ثنائية الأبعاد (2–بسيطة). لقد أظهرنا أنه في ظل هذه الظروف يكون المشغل ثنائي التطابق تناظرًا منزليًا، أي أن محدد جاكوبي الخاص به غير صفري. سبق أن ثبت أن أي مشغل LV الذي يفي بشروط f–متباينة يكون ثنائي التطابق. هنا، نثبت عكسه من خلال إعطاء مثال على مشغل LV ثنائي التطابق ليس f–متباينة. في الجزء الثاني، ندرس السلوك المقارب لفئة من مشغل LV محدد على 2–بسيطة. نجد أن نقطة ثابتة داخلية له فريدة من نوعها. ثم، من خلال بناء دالة لياپانوف، تظهر أن مجموعة الحدود تقع في الحدود. كما نقدم وصفًا للمسار الذي يسلكه مساره عندما يميل عدد التكرارات إلى اللانهاية. نقدر الوقت الذي يقضيه المسار في حي من أحد الرؤوس، ونجد أن المسار لا يبقى في أي من الأحياء. نستمر في إثبات أن المشغل يمتلك خاصية عدم إِرْغَوْدِيّ. و في الجزء التالي، يتم النظر في حالة خاصة من المشغل أعلاه، حيث نعتبر مزيجًا محدبًا لهذا المشغل غير إِرْغَوْدِيّ من الدرجة الأعلى ومشغل LV تربيعي ثنائي التطابق. نجد أن نقاط ثابتة موجودة عند حافة 2–بسيطة عندما يكون معيارها الخاص أعلى من قيمة حرجة. علاوة على ذلك، فهي هياكل حدبة عندما يتجاوز المعلمة القيمة الحرجة بكمية معينة أو تنفر إذا لم يكن كذلك. ثم، من خلال فرض شرط على معياره، نقوم ببناء دالة لياپانوف للمشغل. تظهر أنه، في ظل مثل هذه الحالة؛ يمتلك المشغل LV المشتق من مثل هذا المزيج خاصية الانتظام، ويصل مساره إلى أحد الرؤوس لأي نقطة أولية أخرى غير نقاط الثابتة على الحافة ونقطة الثابتة الداخلية. يتم

الحصول أيضًا على وصف كامل لمجموعة الحدود لمثل هذا المسار. في الجزء الأخير، نعتبر فئة من مشغل LV محدد على 3-بسيطة (3-بسيطة). بالإضافة إلى وجود نقطة ثابتة داخلية فريدة من نوعها، نظهر أيضًا وجود عدد لا حصر له من نقاط ثابتة على اثنتين من حوافه. نقوم ببناء العديد من وظائف لياپانوف، والتي من خلالها نجد أن مساره يقترب من الحواف. ثم نثبت وجود مجموعة من 3-بسيطة ذات قياس موجب بحيث لا يقترب مسار المشغل لأي نقطة أولية مأخوذة من المجموعة، ويتم وصف المسار النهائي الذي يسلكه مثل هذا المسار. باستخدام تقنية مماثلة نظهر أن متوسط مساره ينحرف، أي أن المشغل غير إِرْغَوْدِيّ. في النهاية، يتم دراسة الديناميكيات على وجوهها ثنائية الأبعاد (2-وجوه). تم العثور على المشغل ليكون منتظمًا عند تقييده على 2-وجوه، وتم تقدير مجموعة الحدود.

APPROVAL PAGE

The thesis of Azizi Bin Rosli has been approved by the following:

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Pah Chin Hee
Supervisor

____

Farrukh Mukhamedov
Co-supervisor

____

Abdumalik Rakhimov
Internal Examiner

____

Uygun Jamilov
External Examiner

____

Azlina Bt. Daud
Chairman

DECLARATION

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This thesis is dedicated to my family and those who have taken part in its completion.

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LIST OF SYMBOLS

$\mathbf{c}$	Centre of a simplex
∂S^{m-1}	Boundary of S^{m-1}
$\mathbf{e}_i$	Standard basis of S^{m-1}
E	m-tuple
$Fix(V)$	A set of fixed points of operator V
$\varphi(\mathbf{x})$	Lyapunov Function of a simplex
G_β	Subset of a simplex
Γ_α	$(\lvert\alpha\rvert - 1)$-dimensional faces of S^{m-1}
I	Unit segment
$int(A)$	Interior of a set A
J	Jacobi Matrix
$\lvert J\rvert$	Jacobian
$\mathbb{N}$	Natural Numbers
$\omega(\mathbf{x})$	Omega-limiting set
$p_{ijk,l}$	Heredity coefficient
π	Permutation
$\mathbb{R}$	Real Numbers
$\mathbb{R}^m$	m-dimensional Real Space
S^{m-1}	$(m-1)$-dimensional simplex
θ, γ_i	Parameter
U, V, W	Stochastic Operator
$\mathbb{U}$	Neighbourhood of a vertex
$\lVert\mathbf{x}\rVert$	Norm
$\langle\mathbf{x},\mathbf{y}\rangle$	Scalar product

LIST OF ABBREVIATIONS

AM	Arithmetic mean
CSO	Cubic Stochastic Operator
GM	Geometric mean
LHS	Left-hand side
LV	Lotka-Volterra
QSO	Quadratic Stochastic Operator
RHS	Right-hand side

CHAPTER ONE

INTRODUCTION

1.1 LITERATURE REVIEW

Non-linear Markov operator was proven to be a rich source of tools for modelling and analysis in various field of science. To name a few, it has some applications in evolutionary biology, population dynamics, epidemiology, and game theory (Kolokoltsov, 2010). Among the simplest but widely studied discrete time non-linear Markov operator is known as quadratic stochastic operator (QSO), initiated and introduced by Bernstein (1942) in his study of theory of heredity. Since then, it becomes a primary source of investigation in population dynamics (Hofbauer & Sigmund, 1998; Y. Lyubich, 1994; F. Mukhamedov & Saburov, 2012). Besides having extensive application in biology, for example in modelling the Mendelian and non-Mendelian inheritances of a bisexual population (N. Ganikhodjaev et al., 2013) and the evolution of blood groups and rhesus factor of a population (N. Ganikhodjaev et al., 2010; Saburov & Arshat, 2017); it is also known that non-linear Markov operator has many applications in physics (Plank, 1995; Takeuchi, 1996; Udwadia & Raju, 1998) as well as mathematics (Kesten, 1970; S. M. Ulam, 1964). A more recent studies showed that QSOs also have its application in investigating the ocean ecosystem (Rozikov & Shoyimardonov, 2019) and modelling the spread of COVID-19 disease (Rozikov & Shoyimardonov, 2021).

Despite being the simplest, the complexity in studying a QSO lies in the cubic stochastic matrix associated with it. Therefore, several classes of QSOs, the most well-known being Volterra QSO; were introduced in studying the asymptotic behaviour of its trajectory – one of the main problems in non-linear operator theory (N. Ganikhodjaev & Jusoo, 2018; Hardin & Rozikov, 2018; U. Jamilov, 2016; F. Mukhamedov, Embong, & Pah, 2017; F. Mukhamedov & Embong, 2015; Shahidi, 2008; Zhamilov & Rozikov, 2009).

Based on previous studies, a non-linear stochastic operator, in particular QSO; could be defined on either a state space or a simplex. However, even though the theory of QSO defined on a state-space is also well developed and interesting to explore (N. Ganikhodjaev, 2016), this thesis will be focusing only on non-linear operator defined on a simplex.

Regardless of all the classes of QSOs introduced, however; the dynamics of a QSO in general especially non-Volterra QSO is still an open problem (R. Ganikhodzhaev et al., 2011; U. U. Jamilov & Khudoyberdiev, 2023; Khamrayev, 2020). Up to date, only QSO defined on 1-dimensional simplex was fully studied by Lyubich (Y. I. Lyubich, 1992), in which he showed that the ω-limit set, as we would define later; of any initial point is a finite set. Naturally, some classes of cubic stochastic operators (CSO) were also introduced (Davronov et al., 2015; U. U. Jamilov et al., 2018; U. U. Jamilov & Reinfelds, 2020a; Rozikov & Khamraev, 2004).

It is good to note that both Volterra QSO and CSO belong to a class or family of stochastic operator known as the Lotka-Volterra (LV) operator. Its continuous-time analogue, referred as the Lotka-Volterra systems were first used by Lotka (Lotka, 1925) and Volterra (Volterra, 1926) to model the time evolution of conflicting species in biology – also known as the predator-prey model. After its introduction in biomathematical context by Moran (Moran, 1950), the application of discrete-time models of LV systems were expanded to include different areas of natural sciences (Basson & Fogarty, 1997; Dohtani, 1992; Fisher & Goh, 1977; Goel et al., 1971; Hofbauer et al., 1987; Lu & Wang, 1999). Not limiting to quadratic LV systems, cubic LV systems such as the Lotka-Volterra-Brusselator model (Farkas et al., 1989), as well as higher order LV systems were also introduced and studied (Goel et al., 1971; Safuan et al., 2013).

Previously, several classes of discrete-time LV systems such as the LV operator were investigated in the form of Volterra QSO and CSO. Historically, the simplest quadratic LV operator acting on 2-dimensional simplex considered by Zakharevich (1978) was shown to be non-ergodic, disproving Ulam's (1960) conjecture that ergodic theorem holds for any QSO defined on a finite-dimensional simplex. Zakharevich's example was then generalized by N. Ganikhodjaev and D. Zanin, and its necessary condition for non-ergodicity was given (2004). Later, several examples of non-ergodic quadratic LV operators acting on higher dimensional simplex were also studied (N. N.

Ganikhodjaev et al., 2013, 2015). The problem was generalized further by Saburov to include higher order LV operator as well (2015). Then based on previous work on non-ergodic LV operator, conditions for non-ergodic cubic LV operator was constructed by Mukhamedov et al. (2019).

Since an LV operator is a self-mapping of a compact convex set, Brouwer's fixed point theorem guarantees at least one fixed point. It is well known that if a trajectory converges, it must converge to a fixed point or an attractor, but convergence is not always the case as mentioned earlier. Hence, a Lyapunov function is often constructed to study the dynamics of an LV operator. Even though there is no general rule of how a Lyapunov function could be generated for any given operator, it is often used to obtain the upper estimate of an ω-limit set. Unlike the Jacobian, a linearization method which only describes the local behaviour of an operator near a fixed point; Lyapunov function could be used to get a wider view of the global behaviour. For example, the Lyapunov method was used to describe the dynamics of a quadratic LV operator defined on a finite-dimensional simplex by Ganikhodjaev (1993), an extension to the quadratic operator studied by Vallander (1972).

Many Lyapunov functions, both linear and non-linear; have been constructed for different classes of LV operators defined on a simplex (R. N. Ganikhodzhaev & Saburov, 2008; U. U. Jamilov, 2012; F. Mukhamedov & Embong, 2017). However, despite being a tool to obtain the upper estimate of ω-limit set; it has no use in determining the fractal dimension of an infinite ω-limit set. Take Zakharevich's example (1978), it was known that ω-limit set of any interior point of 2-dimensional simplex except the centre is infinite, and it lies in the boundary of the simplex. What was not known was whether the ω-limit set forms a heteroclinic cycle equals to the boundary of the simplex (Vallander, 1972). The study on its fractal dimension was done many years later by Baranski and Misiurewicz (2009) in which the authors were answering the same question posed by Vallander.

We say that an LV operator is regular if, for any initial point; its trajectory converges, i.e., the ω-limit set is singular. Additionally, if the ω-limit set is found to be finite, then its respective trajectory is asymptotically periodic. However, it is often the case that ω-limit set is infinite for previously studied non-ergodic LV operator. Keep in mind that a regular LV operator is also an ergodic operator, but the inverse argument may not hold. Thus, existence of a diverging trajectory does not directly imply non-ergodicity.

Besides the limiting behaviour of the trajectory of an LV operator, numerous studies were also done on its properties. It was proven that a quadratic LV operator acting on a finite simplex is a homeomorphism (R. N. Ganikhodzhaev, 1993). In fact, a QSO in general was shown to be a surjection if and only if it is a bijection on its simplex (Saburov, 2016). In the same paper, it was also established that any order LV operator is a surjection. Further study on surjective property was also done for a CSO, in which the authors showed that a CSO is a surjection if and only if it is orthogonal preserving (F. Mukhamedov, Embong, & Rosli, 2017). Note that both quadratic and cubic LV operator were shown to be orthogonal preserving (Akın & Mukhamedov, 2015; F. Mukhamedov, Embong, & Pah, 2017). As reference, some examples and constructions of orthogonal preserving QSO and CSO were done for 2-dimensional simplex (F. Mukhamedov, Embong, & Rosli, 2017; F. Mukhamedov & Taha, 2016).

Based on previous studies, one may suggest that any LV operator is also a bijection. However, it is not always the case as shown in few examples of non-bijective linear LV operator given by Mukhamedov and Saburov (2017). The authors also introduced the notion of $\boldsymbol{f}$-monotone, concluding that any $\boldsymbol{f}$-monotone type LV operator acting on finite dimensional simplex is a homeomorphism of its simplex.

1.2 PROBLEM STATEMENT

It is evident from previous findings that studying the dynamics of a non-linear stochastic operator is challenging even for a lower order operator defined on lower dimensional simplex. This fact could be seen from many previous operators investigated especially on non-Volterra type operator such as b-bistochastic QSO (F. Mukhamedov & Embong, 2015), dissipative QSO (Shahidi, 2008), separable QSO (Rozikov & Nazir, 2010), and conditional CSO (Davronov et al., 2015). Notice how many studies were done on QSO, yet those studies combined are not enough to conclude the dynamics of QSO in general – the same can be said for CSO or any higher order non-linear stochastic operator.

LV operator, for instance; is well developed for the quadratic case (R. N. Ganikhodzhaev, 1993), but there are still many unsolved problems in higher order case even for those defined on lower dimensional simplex. Even for cubic LV operator, only few classes of operator were introduced until recently (U. U. Jamilov et al., 2018; U. U. Jamilov & Reinfelds, 2020; Rozikov & Khamraev, 2004). Hence, introducing and investigating classes of higher order LV operator is an interesting and logical approach (for reference, see Mukhamedov & Saburov (2014), and Saburov (2015), and Mukhamedov et al. (2019)). Nevertheless, studies on higher order LV operator are few and not exhaustive due to the countless way it could be constructed.

It is also interesting to investigate a class of operator generated by a convex combination of two known LV operators. Such study was done by Jamilov and Reinfelds (2021) using two cubic LV operators of differing dynamics. It was shown that phase transition exists, and depending on its parameter it is either regular or non-ergodic. In this thesis, however; we are more interested in studying the dynamics of an LV operator generated by a convex combination of two LV operators of different order with opposite dynamics. It is not yet known how the skewness in the order would affect the dynamics.

Recall that any LV operator is a surjection. Its bijectivity however, was only proven for some classes of LV operator, e.g., quadratic LV operator (R. N. Ganikhodzhaev, 1993) and **f**-monotone LV operator (F. Mukhamedov & Saburov, 2017). It was shown that any **f**-monotone LV operator is a homeomorphism, but it is not known if a homeomorphic LV operator would also be **f**-monotone. Thus, it is fascinating to find an LV operator which is a homeomorphism but not **f**-monotone.

1.3 RESEARH OBJECTIVES

The objectives of this research are:

i. to determine the bijectivity of a class of Lotka-Volterra operator defined on the 2-dimensional simplex by finding its condition for homeomorphism;
ii. to study the limiting behaviour of a class of Lotka-Volterra operator defined on the 2-dimensional simplex using Lyapunov function;

iii. to study the dynamics of a class of Lotka-Volterra operator generated by a convex combination of a non-ergodic higher order Lotka-Volterra operator and a regular quadratic Lotka-Volterra operator using Lyapunov function;

iv. to study the limiting behaviour of a class of Lotka-Volterra operator defined on a 3-dimensional simplex using Lyapunov function.

1.4 THESIS ORGANIZATION

This thesis is organized into seven chapters. The first chapter is dedicated to the literature review on non-linear stochastic operator, primarily on backgrounds and problems related to the Lotka-Volterra operator.

In Chapter 2, we recall some definitions related to non-linear stochastic operator defined on a finite-dimensional simplex. Those definitions also apply to Lotka-Volterra operator which is defined afterward. Later, we recall some notions on surjectivity and **f**-monotonicity of the operator. Some results and examples on operators generated by a convex combination of two Lotka-Volterra operators are also given. At the end of the chapter, we introduce a general class of operator which our study in the subsequent chapters will be based on. To be clear, the next three chapters will be on Lotka-Volterra operator defined on the 2-dimensional simplex, and the sixth chapter will be on Lotka-Volterra operator defined on the 3-dimensional simplex.

We devote Chapter 3 to studying the bijectivity of a class of Lotka-Volterra operator defined on the 2-dimensional simplex. Several cases are considered, and we summarise its sufficient conditions of bijection in the main theorem. The aim is clear, we provide conditions under which the operator has a non-zero Jacobian, i.e., the operator is a homeomorphism. We refer Ganikhodjaev (1993) for the methodology used in this chapter. Some examples of bijective Lotka-Volterra operator are given, and its relationship with **f**-monotonicity is examined. We will also have a look at its application in finding a solution to a Hammerstein integral equation.

We begin our study on the dynamics of a class of Lotka-Volterra operator in Chapter 4. Here, we consider a class of Lotka-Volterra operator defined on the 2-

dimensional simplex. Besides its fixed point, we also construct a Lyapunov function to estimate its omega-limiting set. Using similar technique to Zakharevich (1978), we provide the eventual path taken by its trajectory, and the time it spends inside a neighbourhood of a vertex is also estimated. Using these facts, we constructed a sequence to represent such trajectory, and the Cesaro summability of the trajectory is investigated. Note that this class of operator covers many non-ergodic Lotka-Volterra operators studied in the past as shown in Chapter 2.

In Chapter 5, we consider a special case of non-ergodic operator from the previous chapter. Then, we form an operator from the convex combination of the operator and a class of quadratic Lotka-Volterra operator. Note that the quadratic component of the operator was proven to be regular (U. Jamilov & Reinfelds, 2021). Some restriction on its parameter will be considered accordingly as we move deeper in our investigation. We find its fixed points, and the stability of those fixed points on the boundary of the simplex are studied. Due to its complexity, we consider a stricter condition on its parameter to construct a Lyapunov function. Together with some auxiliary findings on the limiting behaviour of its trajectory, we provide a complete description of its omega-limiting set by the end of the chapter.

We consider a class of Lotka-Volterra operator defined on the 3-dimensional simplex in Chapter 6. To provide a richer result on our analysis, notably on our auxiliary results on the dynamics of its trajectory on the 2-dimensional faces of the simplex; this class of operator is more restricted than the one in Chapter 4. The methodology and calculation used in this chapter is similar to Chapter 4. First, we find its fixed points and investigate its stability. Then, we construct several Lyapunov functions to estimate its omega-limiting set. For a subset of the simplex, we describe the path taken by its trajectory as time tends to infinity. Also, the time it spends in a neighbourhood of a vertex is estimated. The sequence of such trajectory is constructed, and its Cesaro summability is then investigated. In the end of the chapter, we provide the results of our analysis on the dynamics of the operator restricted on one of the 2-dimensional faces of the simplex.

CHAPTER TWO

LOTKA-VOLTERRA OPERATOR

Before we go further on the dynamics and properties of a Lotka-Volterra operator, it is always good to introduce the notion of a stochastic operator defined on a simplex. Let $E = \{1,2,\dots,m\}$ be an m-tuple, then an $(m-1)$-dimensional simplex is a collection of probability distributions given by

$$S^{m-1} = \left\{ \mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m \,\middle|\, x_i \geq 0, \forall i \in E \text{ and } \sum_{i \in E} x_i = 1 \right\}.$$

In biological setting, one may take E as a collection of interacting species in a population, and x_i be the relative frequency of a species in the population.

2.1 NON-LINEAR STOCHASTIC OPERATOR

In general, a d-dimensional stochastic operator defined on an $(m-1)$-dimensional simplex is a self-mapping $V: S^{m-1} \to S^{m-1}$ defined by

$$V(\mathbf{x})_l = \sum_{a_1,\dots,a_d \in E} P(l|a_1 a_2 \cdots a_d) x_{a_1} x_{a_2} \cdots x_{a_d}, \qquad l \in E, \tag{2.1.1}$$

where $P(l|a_1 a_2 \cdots a_d)$, which we later write as $p_{a_1 a_2 \cdots a_d, l}$; is the conditional probability of getting l given $a_1,\dots,a_d$, and it satisfies the following properties (Scheutzow & Wilke-Berenguer, 2018):

i. $p_{a_1 a_2 \cdots a_d, l} \geq 0$;
ii. $\sum_{l \in E} p_{a_1 a_2 \cdots a_d, l} = 1$;
iii. $p_{a_1 a_2 \cdots a_d, l}$ is equal for any permutation of $a_1, a_2, \cdots, a_d$.

Moreover, if $l \notin \{a_1, a_2, \dots, a_d\} \Rightarrow p_{a_1 a_2 \cdots a_d, l} = 0$, the corresponding operator is known to be a Volterra type stochastic operator. The biological interpretation of the property is clear, the offspring only inherits the genotype of its parents. Note that $p_{a_1 a_2 \cdots a_d, l}$ is also known as probability or heredity coefficient.

Let $\mathbf{x}^{(t)} = V^{(t)}(\mathbf{x}), t = 0,1,2, \ldots$, the mapping V forms a dynamical system

$$\mathbf{x}^{(0)}, \mathbf{x}', \mathbf{x}'', \mathbf{x}^{(3)}, \ldots$$

also known as the trajectory or orbit of V. The mapping V is also known as the evolution operator since it maps the distribution corresponding to an initial state t to the next state $t+1$, i.e., $V(\mathbf{x}^{(t)}) = \mathbf{x}^{(t+1)}$. Additionally, an element or subject i is said to vanish or extinct if $\lim_{t\to\infty} x_i^{(t)} = 0$. Otherwise, the element i is said to persist. If the mapping V is invertible, for example when V is a bijection; then its negative orbit,

$$\mathbf{x}^{(0)}, \mathbf{x}^{(-1)}, \mathbf{x}^{(-2)}, \ldots$$

also exists.

Throughout this thesis, we denote the vertices of an $(m-1)$-dimensional simplex as $\mathbf{e}_i$, the standard basis of S^{m-1}. Also, the following notations are used to address some subsets of the simplex, S^{m-1}:

i. the set

$$\partial S^{m-1} = \left\{ \mathbf{x} \in S^{m-1} \,\middle|\, \prod_{i\in E} x_i = 0 \right\}$$

is the boundary of the simplex;

ii. the set

$$\operatorname{int}(S^{m-1}) = \left\{ \mathbf{x} \in S^{m-1} \,\middle|\, \prod_{i\in E} x_i > 0 \right\}$$

is the interior of the simplex;

iii. the set

$$\Gamma_\alpha = \{\mathbf{x} \in S^{m-1} | x_i = 0, \forall i \notin \alpha\} = conv(\mathbf{e}_i)_{i\in\alpha}$$

where $\alpha \subset E$ is the $(|\alpha| - 1)$-dimensional face of the simplex, and $\text{conv}(\mathbf{e}_i)_{i\in\alpha}$ is the convex hull of the set $\{\mathbf{e}_i\}_{i\in\alpha}$;

iv. the set

$$\text{int}(\Gamma_\alpha) = \left\{\mathbf{x} \in \Gamma_\alpha \middle| \prod_{i\in\alpha} x_i > 0\right\}$$

is the interior of the face Γ_α.

For simplicity, we may omit notations other than numbers from the index of a set. As example, suppose we have a set Γ_α, where $\alpha = \{1,2\}$. Instead of $\Gamma_{\{1\}}$ and $\Gamma_{\{1,2\}}$, we may write as Γ_1 and Γ_{12}. Note that we may also define the interior of the simplex as $\text{int}(S^{m-1}) = S^{m-1} \setminus \partial S^{m-1}$.

Let $Fix(V)$ be the set of fixed points of V. Since V maps a compact convex set to itself, Brouwer's fixed-point theorem guarantee the existence of at least one fixed point, i.e. $Fix(V) \neq \emptyset$. Hence, in studying the dynamics of the operator V the following definitions related to its fixed points are useful:

Definition 2.1 (Devaney, 2003) The point $\mathbf{x}^{\text{p}} \in S^{m-1}$ is a fixed point for V if $V(\mathbf{x}^{\text{p}}) = \mathbf{x}^{\text{p}}$.

Definition 2.2 (Devaney, 2003) Let $\mathbf{x} \in S^{m-1}$ and

$$J_V(\mathbf{x}) = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \cdots & \frac{\partial x_m'}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1'}{\partial x_m} & \cdots & \frac{\partial x_m'}{\partial x_m} \end{bmatrix}$$

be the Jacobi matrix of V at a point $\mathbf{x}$, where x_i' is a component of $V(\mathbf{x})$. Suppose $\lambda_1, \lambda_2, \dots, \lambda_m$ are the eigenvalues of J_V at a fixed point $\mathbf{x}^{\mathrm{p}}$, then the fixed point $\mathbf{x}^{\mathrm{p}}$ is a hyperbolic fixed point if $|\lambda_i| \neq 1$ for all $i \in E$. Furthermore, a hyperbolic fixed point $\mathbf{x}^{\mathrm{p}}$ is

i. attracting or stable if $|\lambda_i| < 1$ for all $i \in E$;
ii. repelling or unstable if $|\lambda_i| > 1$ for all $i \in E$;
iii. a saddle-node if otherwise.

Definition 2.3 (Devaney, 2003) The point $\mathbf{x}^* \in S^{m-1}$ is called a periodic point of period k for V if $V^{(k)}(\mathbf{x}^*) = \mathbf{x}^*$. If $k = 1$, then $\mathbf{x}^*$ is a fixed point.

In some literature, an attracting fixed point, $\mathbf{x}^{\mathrm{p}}$ is a type of attractor, and a set $B(\mathbf{x}^{\mathrm{p}}) \subset S^{m-1}$ is called the basin of attraction for the attractor if $\lim_{t\to\infty} V^{(t)}\left(\mathbf{x}^{(0)}\right) = \mathbf{x}^{\mathrm{p}}$ for all $\mathbf{x}^{(0)} \in B(\mathbf{x}^{\mathrm{p}})$.

However, studying only the fixed points has its own limitation. It describes the local behaviour near the point but gives a vague idea on the global behaviour of the operator. For example, existence of an attracting fixed point, $\mathbf{x}^{\mathrm{p}}$ does not immediately imply that the trajectory of any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ will converge to $\mathbf{x}^{\mathrm{p}}$ under V.

For such reason, a Lyapunov function may be constructed as a method to get the upper estimate of the omega-limit set, $\omega\left(\mathbf{x}^{(0)}\right)$ of an initial point $\mathbf{x}^{(0)} \in S^{m-1}$ with respect to the operator V. By upper estimate we mean the smallest set containing $\omega\left(\mathbf{x}^{(0)}\right)$.

Definition 2.4 A continuous function $\varphi(\mathbf{x}): S^{m-1} \to \mathbb{R}$ is called a Lyapunov function of V if the limit $\lim_{t\to\infty} \varphi\left(V^{(t)}(\mathbf{x})\right)$ exists for any $\mathbf{x} \in S^{m-1}$.

Definition 2.5 Let $\mathbf{x}^{(0)} \in S^{m-1}$, then the omega-limit set of $\mathbf{x}^{(0)}$ is defined as

$$\omega\left(\mathbf{x}^{(0)}\right) = \bigcap_{n=0}^{\infty} \overline{\{\mathbf{x}^{(k)} | k \geq n\}}.$$

Suppose $\lim_{t\to\infty} \varphi\left(V^{(t)}(\mathbf{x})\right) = c$ be the limit of a Lyapunov function $\varphi(\mathbf{x})$, then we have $\omega(\mathbf{x}) \subset \varphi^{-1}(\mathbf{c})$. For example, consider $\varphi(\mathbf{x}) = x_1x_2x_3, \mathbf{x} = (x_1, x_2, x_3) \in S^2$. Suppose $\lim_{t\to\infty} \varphi\left(V^{(t)}(\mathbf{x})\right) = 0$, then since $x_1x_2x_3 = 0 \Leftrightarrow \mathbf{x} \in \partial S^2$ we have $\omega(\mathbf{x}) \subset \partial S^2$.

Regularity of an operator could be determined by studying the omega-limit sets of each initial point in a simplex. If the omega-limit sets are singular, then the operator is regular. Regularity of an operator V is defined as follows:

Definition 2.6 An operator V is called regular if for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$, the limit

$$\lim_{t\to\infty} V^{(t)}(\mathbf{x}) \tag{2.1.2}$$

exists.

Ergodicity, on the other hand; is defined by the Cesaro summability of the sequence $\left\{\mathbf{x}^{(t)}\right\}_{t=0}^{\infty}$ as followings:

Definition 2.7 An operator V is ergodic if for any initial point $x^{(0)} \in S^{m-1}$, the limit

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} V^{(k)}(\mathbf{x}) \tag{2.1.3}$$

exists.

Clearly, a regular operator V is also ergodic. Contrapositively, a non-ergodic operator V is also non-regular.

For a non-ergodic operator, it is often the case that the omega-limit sets are infinite for some initial points in the simplex. Let $X \subset S^{m-1}$ and $\mu(X) > 0$, where μ is the usual Lebesgue measure, then an operator V is non-ergodic if for any $\mathbf{x} \in X$, the limit $\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} V^k(\mathbf{x})$ diverges.

In addition to the dynamics of V, some studies were also done on its properties, notably on surjectivity and bijectivity of the mapping. One way of showing that a mapping V is a bijection is by proving that V is a homeomorphism. Since all the faces of S^{m-1} are invariant, V maps each face of S^{m-1} homeomorphically, assuming that it is; onto itself by induction. Hence, by the compactness of S^{m-1} it is sufficient to show that the mapping V is a local homeomorphism at each $\mathbf{x} \in int(S^{m-1})$.

In some cases where induction fails or inapplicable, we may show that V maps any faces of S^{m-1} homeomorphically to itself directly. Let $\bar{V}_{\Gamma,\alpha}: S^{m-1-|\alpha|} \to S^{m-1-|\alpha|}$ be the restriction of V on each face $\Gamma_\alpha, \alpha \subset E$. We say that V maps any faces of S^{m-1} homeomorphically onto itself if the determinant of the Jacobi matrix of $\bar{V}_{\Gamma,\alpha}, J_{\bar{V}_{\Gamma,\alpha}}$ is non-zero for any $\alpha \subset E$, i.e., $\left|J_{\bar{V}_{\Gamma,\alpha}}\right| \neq 0$.

The simplest example of a non-linear stochastic operator would be the quadratic stochastic operator, $V_{quad}: S^{m-1} \to S^{m-1}$ defined by

$$V_{quad}(\mathbf{x})_k = \sum_{i,j\in E} p_{ij,k} x_i x_j, \qquad k \in E, \tag{2.1.4}$$

where $p_{ij,k} \geq 0, \sum_{k\in E} p_{ij,k} = 1$, and $p_{ij,k} = p_{ji,k}$.

Similarly, a cubic stochastic operator is a mapping $V_{cube}: S^{m-1} \to S^{m-1}$ defined by

$$V_{cube}(\mathbf{x})_l = \sum_{i,j,k\in E} p_{ijk,l} x_i x_j x_k, \qquad l \in E, \tag{2.1.5}$$

where $p_{ijk,l} \geq 0, \sum_{l\in E} p_{ijk,l} = 1$, and $p_{ijk,l} = p_{jik,l} = p_{jki,l} = p_{ikj,l} = p_{kij,l} = p_{kji,l}$.

Both quadratic and cubic stochastic operators have been the subjects of many studies on the dynamics of a non-linear stochastic operator since 1942 (Bernstein, 1942). Despite QSOs being extensively studied in the past 80 years, more classes of QSOs are still being introduced. To get a view of how many classes of QSOs could be introduced, we may have a look at QSOs generated by a 2-dimensional simplex, S^2. Since $x_1 + x_2 + x_3 = 1$, we have $(x_1 + x_2 + x_3)^r = 1$ for any $r \in \mathbb{N}$. Taking $r = 2$ gives us

$$(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 1.$$

Distributing above terms into x_1', x_2', and x_3' differently will get us different operators which some of them are invariant under permutation. In Zakharevich (1978), for example; one of such operators given by

$$V_{zakh}(\mathbf{x}) = \begin{cases} x_1' = x_1^2 + 2x_1x_2, \\ x_2' = x_2^2 + 2x_2x_3, \\ x_3' = x_3^2 + 2x_3x_1 \end{cases} \tag{2.1.6}$$

were studied. As counter example to Ulam's conjecture on ergodicity of any QSO on S^{m-1}, operator (2.1.6) was shown to be non-ergodic.

In fact, any of those terms could be written as sums of another terms, e.g., $2x_1x_2 = (1 + a)x_1x_2 + (1 - a)x_1x_2, a \in [-1,1]$. If we do the same for $2x_2x_3$ with parameter b and $2x_3x_1$ with parameter c, we may introduce the operator

$$V_{zakh,2}(\mathbf{x}) = \begin{cases} x_1' = x_1^2 + (1 + a)x_1x_2 + (1 - c)x_3x_1, \\ x_2' = x_2^2 + (1 + b)x_2x_3 + (1 - a)x_1x_2, \\ x_3' = x_3^2 + (1 + c)x_3x_1 + (1 - b)x_2x_3, \end{cases} \tag{2.1.7}$$

where $a, b, c \in [-1,1]$. If $a, b, c = 1$, then equation (2.1.7) is reduced to (2.1.6). This operator was proven to be non-ergodic when $a, b,$ and c have the same sign by Ganikhodjaev and Zanin (2004).

A more general operator could be introduced by substituting $2x_1x_2 = \left(1 + a\xi(x_1)f(\mathbf{x})\right)x_1x_2 + \left(1 - a\xi(x_1)f(\mathbf{x})\right)x_1x_2$. If similar substitution is done to $2x_2x_3$ and $2x_3x_1$ using expressions $b\xi(x_2)$ and $c\xi(x_3)$, respectively we will obtain

$$V_\xi(\mathbf{x}) = \begin{cases} x_1' = x_1^2 + \left(1 + a\xi(x_1)f(\mathbf{x})\right)x_1x_2 + \left(1 - c\xi(x_3)f(\mathbf{x})\right)x_3x_1, \\ x_2' = x_2^2 + \left(1 + b\xi(x_2)f(\mathbf{x})\right)x_2x_3 + \left(1 - a\xi(x_1)f(\mathbf{x})\right)x_1x_2, \\ x_3' = x_3^2 + \left(1 + c\xi(x_3)f(\mathbf{x})\right)x_3x_1 + \left(1 - b\xi(x_2)f(\mathbf{x})\right)x_2x_3, \end{cases}$$

where $a, b, c \in [-1,1], \xi(x): [0,1] \to [0,1]$, and $f(\mathbf{x}): S^2 \to [0,1]$.

Since $\mathbf{x} \in S^{m-1}$, we can write the above operator into a simpler form by factoring x_1 from x_1' and substituting $x_1 = 1 - x_2 - x_3$ into the multiplicand. If we do similarly for x_2' and x_3' we may rewrite the operator as

$$V_\xi(\mathbf{x}) = \begin{cases} x_1[1 + (a\xi(x_1)x_2 - c\xi(x_3)x_3)f(\mathbf{x})], \\ x_2[1 + (b\xi(x_2)x_3 - a\xi(x_1)x_1)f(\mathbf{x})], \\ x_3[1 + (c\xi(x_3)x_1 - b\xi(x_2)x_2)f(\mathbf{x})]. \end{cases} \qquad (2.1.8)$$

If $\xi(x)$ is a power function, the degree of $\xi(x)$ will determine the dimension of V_ξ, e.g. cubic when the degree of $\xi(x)$ is one, or quadratic when it is a constant as studied by Saburov (2015). This operator, as we would discuss later in more detail; is one of many classes of LV operator.

2.2 LOTKA-VOLTERRA OPERATOR

Consider a mapping $\mathbf{f}: S^{m-1} \to \left(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})\right) \in \mathbb{R}^m$, we define a Lotka-Volterra operator as follows:

Theorem 2.8 (F. Mukhamedov & M. Saburov, 2014) A mapping $U: S^{m-1} \to S^{m-1}$ defined by

$$U(\mathbf{x})_k = x_k\left(1 + f_k(\mathbf{x})\right), \qquad k \in E, \qquad (2.2.1)$$

is a Lotka-Volterra operator if and only if $\mathbf{f}: S^{m-1} \to \mathbb{R}^m$ satisfies the following conditions:

i. $\mathbf{f}$ is continuous in S^{m-1};

ii. for every $\mathbf{x} \in S^{m-1}$, one has $f_k(\mathbf{x}) \geq -1$ for all $k \in E$;

iii. for every $\mathbf{x} \in S^{m-1}$, one has $\sum_{k=1}^{m} x_k f_k(\mathbf{x}) = 0$;

iv. for every $\alpha \subset E$, one has $f_k(\mathbf{x}) > -1$ for all $\mathbf{x} \in int(\Gamma_\alpha),\ k \in \alpha$.

In short, if U is an LV operator, then each face of S^{m-1} is invariant with respect to U, i.e., $U(\Gamma_\alpha) \subset \Gamma_\alpha$ for any $\alpha \subset E$.

It is clear from Theorem 2.8 that operator V_ξ defined by (2.1.8) is an LV type operator. Similarly, it was shown that Volterra QSO and CSO are also LV type operators. This is evident from the fact that any Volterra QSO V_{qso} could be written as

$$V_{qso}(\mathbf{x})_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right), \qquad k \in E, \tag{2.2.2}$$

where $a_{ki} = 1 - 2p_{ik,k}$ for $i \neq k$ and $a_{kk} = 0$ (R. N. Ganikhodzhaev, 1993).

Moreover, any Volterra CSO V_{cso} is equivalent to

$$V_{cso}(\mathbf{x})_l = x_l\left(x_l^2 + x_l\sum_{i\in E\backslash\{l\}} 3p_{ill,l}x_i + \sum_{i,j\in E\backslash\{l\}} 3p_{ijl,l}x_ix_j\right), \qquad l \in E. \tag{2.2.3}$$

We may substitute $x_l = 1 - \sum x_i, i \in E\backslash\{l\}$ and get

$$\begin{aligned} V_{cso}(\mathbf{x})_l &= x_l\left[\left(1 - \sum_{i\in E\backslash\{l]} x_i\right)^2 + \left(1 - \sum_{i\in E\backslash\{l\}} x_i\right)\sum_{i\in E\backslash\{l\}} a_{il}x_i + \sum_{i,j\in E\backslash\{l\}} b_{ijl}x_ix_j\right] \\ &= x_l\left(1 + \sum_{i\in E\backslash\{l\}} a_{il}x_i + \sum_{i,j\in E\backslash\{l\}} b_{ijl}x_ix_j\right) \end{aligned}$$

$$= x_l \left(1 + \sum_{i \in E\setminus\{l\}} a_{il} x_i + \sum_{i \in \setminus E\{l\}} b_{iil} x_i^2 + 2 \sum_{\substack{i,j \in E\setminus\{l\} \\ i<j}} b_{ijl} x_i x_j \right),$$

where $a_{il} = 3p_{ill,l} - 2$ and $b_{ijl} = 1 + 3p_{ijl,l} - 3p_{ill,l}$. Notice it has the form of (2.2.1) and satisfies the properties of Theorem 2.8 due to stochasticity of the CSO.

The QSO (2.2.2) was extensively studied by Ganikhodjaev (1993). It was proven that any Volterra QSO has a decreasing Lyapunov function $\varphi(\mathbf{x}) = x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m}$ for any $\mathbf{x} \in int(S^{m-1})$, where $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in Fix(V_{qso})$. Besides not having a periodic point, it was also shown that the trajectory of any $\mathbf{x} \in int\ (S^2)\setminus\{\mathbf{p}\}$ diverges if V_{qso} has an isolated interior fixed point, $\mathbf{p}$. It is important to note that an operator V does not have a periodic point if $\omega(\mathbf{x})$ is either singular or infinite for any $\mathbf{x} \in int(S^2)\setminus Fix(\mathrm{V})$.

Studies on CSO (2.2.3), on the contrary; are fewer in comparison. Analysis on Volterra CSOs is also more complicated and results are mixed. For instance, consider the mapping $U_\theta: S^{m-1} \to S^{m-1}$

$$U_\theta(\mathbf{x})_l = x_l \left[1 + \left(\sum_{i \in E\setminus\{l\}} x_i - 2 \sum_{i \in E\setminus\{l\}} x_i^2 - 2 \sum_{\substack{i,j \in E\setminus\{l\} \\ i<j}} x_i x_j \right) (3\theta - 2) \right], \qquad \theta \neq \frac{2}{3}$$

studied by Jamilov, Khamraev, and Ladra (2018). Unlike Volterra QSO, even though the operator has an isolated interior fixed point $\mathbf{c} = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$, the trajectory of any point $\mathbf{x} \in S^{m-1}$ converges. It was shown that phase transition occurs at $\theta = \frac{2}{3}$. The fixed point $\mathbf{c}$, for example is attracting when $\theta < \frac{2}{3}$ and repelling when $\theta > \frac{2}{3}$. Similar operator was priorly studied for $\theta = 1$ by Rozikov and Khamraev (2004).

A type of higher order LV operator, so called the M-LV operator was also introduced by Mukhamedov and Saburov (2014), and its trajectory was shown to converge. Denote

$$M(\mathbf{x}) = \left\{ i \in E : x_i = \max_{k \in E} x_k \right\}, \qquad \mathbf{x} \in S^{m-1},$$

then the operator (2.2.1) is an M_1LV operator if for $m \in M(\mathbf{x})$ and $j \in E$ the function $\varphi(\mathbf{x}) = x_m - x_j$ is increasing along the trajectory of any initial point $\mathbf{x} \in S^{m-1}$. Correspondingly, operator (2.2.1) is an M_0LV operator if $\varphi(\mathbf{x})$ is decreasing. By definition, $\varphi(\mathbf{x})$ is a Lyapunov function since $\varphi(\mathbf{x})$ is monotone and bounded by 0 and 1.

For example, as given by the authors; the mapping $V_{\varepsilon,r}: S^{m-1} \to S^{m-1}$ defined by

$$V_{\varepsilon,r}(\mathbf{x})_k = x_k \left[1 + \varepsilon \left(x_k^r - \sum_{i \in E} x_i^{r+1} \right) \right], \qquad k \in E,$$

where $r \in \mathbb{N}$ is an M_1LV operator if $0 < \varepsilon \leq 1$, or M_0LV operator if $-1 \leq \varepsilon < 0$.

Regardless of previous examples on cubic and higher LV operator, convergence is not always the case. One example would be the mapping $W_1: S^2 \to S^2$ defined by

$$W_1(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1 x_2 - x_3^2), \\ x_2' = x_2(1 + x_2 x_3 - x_1^2), \\ x_3' = x_3(1 + x_1 x_3 - x_2^2). \end{cases}$$

This operator was shown to be non-ergodic by Jamilov and Reinfelds (2021). Note that this operator is a reduced case of operator (2.1.8) when $\varphi(x) = x$ and $a = b = c = 1$.

2.3 SURJECTIVITY AND BIJECTIVITY OF LOTKA-VOLTERRA OPERATORS

A mapping $U: S^{m-1} \to S^{m-1}$ is surjective if for any $\mathbf{y} \in S^{m-1}$ we have $\mathbf{x} \in S^{m-1}$ such that $U(\mathbf{x}) = \mathbf{y}$. In general, it was shown by Mukhamedov and Saburov (2017) that any LV operator (2.2.1) is a surjection of S^{m-1} as in the following theorem:

Theorem 2.9 (F. Mukhamedov & M. Saburov, 2017) Any LV type operator given by (2.2.1) maps simplex S^{m-1} onto itself. Namely, U is a surjection of S^{m-1}.

A more detailed study on surjectivity was done by Saburov (2016) for the QSO. It was proven that a QSO defined by (2.1.4) is surjective if and only if it is a permutation of quadratic LV operator defined by (2.2.2). Such permuted operator is known as the π-Volterra operator which is defined as followings:

Definition 2.10 (F. Mukhamedov et al., 2018) A mapping $V: S^{m-1} \to S^{m-1}$ is known as π-Volterra QSO if there is a permutation π of E such that V has the following form:

$$V(\mathbf{x})_k = x_{\pi(k)} \left(1 + \sum_{i \in E} a_{\pi(k)i} x_i\right),$$

where $a_{\pi(k)i} = 2p_{i\pi(k),k} - 1$ and $a_{\pi(k)i} = a_{i\pi(k)}$ for any $i, k \in E$.

Note that any π-Volterra is orthogonal preserving, i.e., for any $\mathbf{x}, \mathbf{y} \in S^{m-1}$ we have $\mathbf{x} \perp \mathbf{y} \Rightarrow V(\mathbf{x}) \perp V(\mathbf{y})$.

In summary, the QSO (2.1.4) is surjective if and only if it is orthogonal preserving or π-Volterra. Further study was done by Mukhamedov et al. (2017) in describing a surjective CSO. It was proven that a surjective CSO is also orthogonal preserving.

Compared to surjectivity, bijectivity on the contrary is not well established for LV operator. While it is well known that any quadratic LV operator defined by (2.2.2) is a homeomorphism of the simplex, i.e., a bijection (R. N. Ganikhodzhaev, 1993), the same cannot be said for LV operator in general.

For example, Mukhamedov and Saburov (2017) provided an example of LV operator defined on S^1 which is not injective (correspondingly, not bijective). In the same paper, the notion of $\mathbf{f}$-monotone LV operator was introduced, and such operator was shown to be a homeomorphism. The following definition describes an $\mathbf{f}$-monotone LV operator:

Definition 2.11 (F. Mukhamedov & Saburov, 2017) An LV operator defined by (2.2.1) is called $\mathbf{f}$-monotone if the mapping $\mathbf{f}: S^{m-1} \to \mathbb{R}^m$ is monotone on S^{m-1}, i.e., for any $\mathbf{x}, \mathbf{y} \in S^{m-1}$ we have

$$\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq 0.$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the scalar product in $\mathbb{R}^m$.

Even though any $\mathbf{f}$-monotone LV operator is a homeomorphism, the converse may not be true. In the upcoming chapter, we will show that a homeomorphic LV operator is not necessarily $\mathbf{f}$-monotone.

2.4 CONVEX COMBINATION OF LOTKA-VOLTERRA OPERATORS

Let $U_1: S^{m-1} \to S^{m-1}$ and $U_2: S^{m-1} \to S^{m-1}$ be two distinct operators, then $U_\theta = \theta U_1 + (1-\theta) U_2, \theta \in [0,1]$ defines a new operator generated by the convex combination of U_1 and U_2. Clearly, the convex combination U_θ is again a self-mapping of the simplex S^{m-1}. Suppose for any $k \in E$ we have the following two LV operators:

$$U_1(\mathbf{x})_k = x_k\big(1 + f_k(\mathbf{x})\big), \qquad U_2(\mathbf{x})_k = x_k\big(1 + g_k(\mathbf{x})\big),$$

where $f_k: S^{m-1} \to \mathbb{R}$ and $g_k: S^{m-1} \to \mathbb{R}$, then the following proposition is true:

Proposition 2.12 For any $\theta \in [0,1]$, the operator $U_\theta = \theta U_1 + (1-\theta)U_2$ is an LV operator.

Proof. From $U_\theta(\mathbf{x})_k$ we obtain

$$\begin{aligned} U_\theta(\mathbf{x})_k &= \theta x_k\big(1 + f_k(\mathbf{x})\big) + (1-\theta)x_k\big(1 + g_k(\mathbf{x})\big) \\ &= x_k\big(1 + \theta f_k(\mathbf{x}) + (1-\theta)g_k(\mathbf{x})\big). \end{aligned}$$

It is easy to check that it satisfies the properties in Theorem 2.8. Consider $\theta f_k(\mathbf{x}) + (1-\theta)g_k(\mathbf{x})$, which is continuous since $f_k(\mathbf{x})$ and $g_k(\mathbf{x})$ are continuous. Due to $f_k(\mathbf{x}), g_k(\mathbf{x}) \geq -1$ we have

$$\theta f_k(\mathbf{x}) + (1-\theta)g_k(\mathbf{x}) \geq -\theta + \theta - 1 \geq -1.$$

Furthermore, since $\sum_{k\in E} x_k f_k(\mathbf{x}) = \sum_{k\in E} x_k g_k(\mathbf{x}) = 0$ we get

$$\sum_{k\in E} x_k[\theta f_k(\mathbf{x}) + (1-\theta)g_k(\mathbf{x})] = \theta \sum_{k\in E} x_k f_k(\mathbf{x}) + (1-\theta)\sum_{k\in E} x_k g_k(\mathbf{x}) = 0.$$

Lastly, for any $A \subset E$ we have $f_k(\mathbf{x}), g_k(\mathbf{x}) > -1$ for all $\mathbf{x} \in int(\Gamma_\alpha),\ k \in \alpha$. Hence

$$\theta f_k(\mathbf{x}) + (1-\theta)g_k(\mathbf{x}) > -\theta + \theta - 1 > -1$$

for all $\mathbf{x} \in int(\Gamma_\alpha),\ k \in \alpha$. This shows that U_θ satisfies Theorem 2.8. □

Consequently, we get the following corollary:

Corollary 2.13 If $f_k(\mathbf{x}) \in [-1,1]$ for any $k \in E$, then the mapping $U_\theta: S^{m-1} \to S^{m-1}$ defined by $U_\theta = \theta U_1 + (1-\theta)U_2$, where

$$U_1(\mathbf{x})_k = x_k\big(1 + f_k(\mathbf{x})\big), \qquad U_2(\mathbf{x})_k = x_k\big(1 - f_k(\mathbf{x})\big),$$

is Lotka-Volterra.

Additionally, in such case where $g_k(\mathbf{x}) = -f_k(\mathbf{x})$ we would have

$$U_\theta(\mathbf{x})_k = x_k\big(1 + (2\theta - 1)f_k(\mathbf{x})\big). \tag{2.4.1}$$

Several studies were done on convex combination of two operators, usually on two operators of different behaviour. For instance, Ganikhodjaev (1989) studied the mapping $V_\lambda: S^2 \to S^2$ defined by $V_\lambda = \lambda V_1 + (1-\lambda)V_2, \lambda \in [0,1]$, where

$$V_1(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_2 - x_3), \\ x_2' = x_2(1 + x_3 - x_1), \\ x_3' = x_3(1 + x_1 - x_2), \end{cases} \qquad V_2(\mathbf{x}) = \begin{cases} x_1' = x_1^2 + 2x_2x_3, \\ x_2' = x_2^2 + 2x_1x_3, \\ x_3' = x_3^2 + 2x_1x_2. \end{cases}$$

Notice that V_1 is an LV operator studied by Zakharevich (1978), while V_2 is not an LV operator. In such case, V_λ is not an LV operator.

Vallander (2013), on the other hand; studied the operator $V_\lambda = \lambda V_1 + (1-\lambda)V_3, \lambda \in [0,1]$, where

$$V_3(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_3 - x_2), \\ x_2' = x_2(1 + x_1 - x_3), \\ x_3' = x_3(1 + x_2 - x_1) \end{cases}$$

is an LV operator. Interestingly, the trajectory of V_λ forms a spiral approaching ∂S^2 for any $\lambda \neq \frac{1}{2}$. The difference between $\lambda < \frac{1}{2}$ and $\lambda > \frac{1}{2}$ lies in the orientation of the spiral, with it resembling the trajectory of V_3 when $\lambda < \frac{1}{2}$. Otherwise if $\lambda > \frac{1}{2}$, the spiral's orientation would resemble V_1.

A more recent study on convex combination of two LV operators were done by Jamilov and Reinfelds (2021). The authors considered a cubic mapping $V_\lambda = \lambda V_4 + (1-\lambda)V_5, \lambda \in [0,1]$, where

$$V_4(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1x_2 - x_3^2), \\ x_2' = x_2(1 + x_2x_3 - x_1^2), \\ x_3' = x_3(1 + x_1x_3 - x_2^2), \end{cases} \qquad V_5(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_3^2 - x_1x_2), \\ x_2' = x_2(1 + x_1^2 - x_2x_3), \\ x_3' = x_3(1 + x_2^2 - x_1x_3). \end{cases}$$

Here, V_4 is a non-ergodic operator, while V_5 is a regular operator. It was shown that V_λ is regular when $\lambda < \frac{1}{2}$, and non-ergodic when $\lambda > \frac{1}{2}$. In the case of $\lambda = \frac{1}{2}$, operator V_λ is an identity mapping.

2.5 $V_{\xi,m-1}$ LOTKA-VOLTERRA OPERATOR

In the upcoming chapters, we consider the mapping $V_{\xi,m-1}: S^{m-1} \to S^{m-1}$ defined by

$$V_{\xi,m-1}(\mathbf{x}) = \begin{cases} x_1' = x_1[1 + (\xi_1(x_1)x_2 - \xi_m(x_m)x_m)f(\mathbf{x})], \\ x_2' = x_2[1 + (\xi_2(x_2)x_3 - \xi_1(x_1)x_1)f(\mathbf{x})], \\ \vdots \\ x_m' = x_m[1 + (\xi_m(x_m)x_1 - \xi_{m-1}(x_{m-1})x_{m-1})f(\mathbf{x})], \end{cases} \tag{2.5.1}$$

where $\xi_i(x): [0,1] \to [-1,1]$ for $i = \overline{1,m}$, and $f(\mathbf{x}): S^{m-1} \to [-1,1]$. Due to complexity of such class of operator, some restrictions on $m, \xi(x)$ and $f(\mathbf{x})$ will be given in each chapter.

Notice that if for any $i \in E$ we have $m = 3$, $f(\mathbf{x}) = 1$, and $\xi_i(x) = 1$, we will recover Zakharevich's operator (2.1.6). Instead, if $m = 3$, $f(\mathbf{x}) = 1$, and $\xi_i(x) = \gamma_i x$, where $\gamma_i \in [-1,1]$, then it will reduce to (2.1.7).

While it is interesting to study $V_{\xi,m-1}$ in general, this thesis, however; will consider 2 and 3-dimensional simplexes only, i.e., $m = 3$ or 4.

CHAPTER THREE

BIJECTIVE LOTKA-VOLTERRA OPERATOR

In this chapter, we consider a class of LV operator defined on S^2. Conditions for its bijectivity are described and some examples are given. Note that if an LV operator is a bijection, then it has an inverse, and its negative trajectory exists. Constructing its inverse, however; would be an independent interest which we will not cover in this chapter.

The methodology by which we prove the bijection of a class of LV operator was given by Ganikhodjaev (1993). In his paper, he proved that Volterra QSO is a homeomorphism by studying the determinant of its Jacobi matrix.

3.1 BIJECTIVITY OF LOTKA-VOLTERRA OPERATOR $V_{\xi,2}$

Recall operator $V_{\xi,m-1}$ from the previous chapter, let $m = 3, f(\mathbf{x}) = 1$, and $\xi_i(x) = \gamma_i \eta_i(x)$, where $\gamma_i \in [-1,1]$ and $\eta_i(x): [0,1] \to [0,1]$ be strictly increasing differentiable convex function, then we have a mapping $V_{\xi,2}: S^2 \to S^2$ defined by

$$V_{\xi,2}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + \gamma_1\eta_1(x_1)x_2 - \gamma_3\eta_3(x_3)x_3), \\ x_2' = x_2(1 + \gamma_2\eta_2(x_2)x_3 - \gamma_1\eta_1(x_1)x_1), \\ x_3' = x_3(1 + \gamma_3\eta_3(x_3)x_1 - \gamma_2\eta_2(x_2)x_2). \end{cases} \qquad (3.1.1)$$

If $\gamma_1 = \gamma_2 = \gamma_3 = 0$, then $V_{\xi,2}$ is an identity mapping. Thus, we consider the case $|\gamma_1| + |\gamma_2| + |\gamma_3| \neq 0$. From $V_{\xi,2}$, we have the following four cases:

i. $\gamma_1, \gamma_2, \gamma_3 \leq 0$;
ii. only one of γ_i is negative;
iii. $\gamma_1, \gamma_2, \gamma_3 \geq 0$;
iv. only two of γ_i are negative.

We will consider each case, with some limitation to the fourth case. Note that the first two cases are equivalent to $\gamma_1\gamma_2\gamma_3 \leq 0$.

Let $J_{V_{\xi,2}}$ be the Jacobi matrix of $V_{\xi,2}$, then to show $V_{\xi,2}$ is a bijection we may show that its Jacobian, $\left|J_{V_{\xi,2}}\right|$ is non-zero for any $\mathbf{x} \in S^2$. Alternatively, we may consider the interior of each face, $\Gamma_\alpha, \alpha \subset \{1,2,3\}$ of S^2.

Let $\mathbf{x} \in int(S^2)$, then from $V_{\xi,2}$ we may obtain the Jacobian $\left|J_{V_{\xi,2}}(\mathbf{x})\right| = x_1x_2x_3|A|$, where

$$A = \begin{bmatrix} \frac{a_{11}}{x_1} & \gamma_1\eta_1(x_1) & -\gamma_3\frac{\partial}{\partial x_3}\eta_3(x_3)x_3 \\ -\gamma_1\frac{\partial}{\partial x_1}\eta_1(x_1)x_1 & \frac{a_{22}}{x_2} & \gamma_2\eta_2(x_2) \\ \gamma_3\eta_3(x_3) & -\gamma_2\frac{\partial}{\partial x_2}\eta_2(x_2)x_2 & \frac{a_{33}}{x_3} \end{bmatrix}, \qquad (3.1.2)$$

and

$$\begin{aligned} a_{11} &= 1 - \gamma_3\eta_3(x_3)x_3 + x_2\gamma_1\frac{\partial}{\partial x_1}\eta_1(x_1)x_1, \\ a_{22} &= 1 - \gamma_1\eta_1(x_1)x_1 + x_3\gamma_2\frac{\partial}{\partial x_2}\eta_2(x_2)x_2, \\ a_{33} &= 1 - \gamma_2\eta_2(x_2)x_2 + x_1\gamma_3\frac{\partial}{\partial x_3}\eta_3(x_3)x_3. \end{aligned} \qquad (3.1.3)$$

Due to Jacobi's theorem, we have $\left|J_{V_{\xi,2}}(\mathbf{x})\right| = 0$ if A is skew-symmetric since $m = 3$ is odd. One could check that A is skew-symmetric if $a_{11} = a_{22} = a_{33} = 0$ or $\eta_i(x) = \frac{\partial}{\partial x}\eta_i(x)x = \eta_i(x) + x\frac{\partial}{\partial x}\eta_i(x)$ for $i = 1,2,3$, i.e., $\mathbf{x} = \emptyset$ which is not in our case.

Furthermore, from A we get

$$
\begin{aligned}
|A(\mathbf{x})| = \frac{a_{11}a_{22}a_{33}}{x_1x_2x_3} &+ \gamma_1\gamma_2\gamma_3\eta_1(x_1)\eta_2(x_2)\eta_3(x_3) \\
&- \gamma_1\gamma_2\gamma_3\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\frac{\partial}{\partial x_2}\eta_2(x_2)x_2\frac{\partial}{\partial x_3}\eta_3(x_3)x_3 \\
&+ \frac{a_{11}}{x_1}\gamma_2^2\eta_2(x_2)\frac{\partial}{\partial x_2}\eta_2(x_2)x_2 + \frac{a_{22}}{x_2}\gamma_3^2\eta_2(x_2)\frac{\partial}{\partial x_3}\eta_3(x_3)x_3 \\
&+ \frac{a_{33}}{x_3}\gamma_1^2\eta_1(x_1)\frac{\partial}{\partial x_1}\eta_1(x_1)x_1.
\end{aligned}
$$

Since $\eta_i(x)$ is strictly increasing, i.e., $\frac{\partial}{\partial x}\eta_i(x) > 0$, we could show that $\eta_i(x) < \eta_i(x) + x\frac{\partial}{\partial x}\eta_i(x) = \frac{\partial}{\partial x}\eta_i(x)x$. Then, the following proposition is clear:

Proposition 3.1 If $a_{11}, a_{22}, a_{33} > 0$ and $\gamma_1\gamma_2\gamma_3 \leq 0$, then $|A| > 0$.

We may also denote $a_{ii} = b_i + C_i$ for $i = 1,2,3$, where

$$
\begin{cases}
C_1 = x_2\gamma_1\dfrac{\partial}{\partial x_1}\eta_1(x_1)x_1, \\
C_2 = x_3\gamma_2\dfrac{\partial}{\partial x_2}\eta_2(x_2)x_2, \\
C_3 = x_1\gamma_3\dfrac{\partial}{\partial x_3}\eta_3(x_3)x_3.
\end{cases}
\qquad (3.1.4)
$$

Clearly $b_i > 0$ and the sign of C_i follows the sign of γ_i, i.e., $sgn(C_i) = sgn(\gamma_i)$. Moreover, if $a_{ii} > 0$ at any point $\mathbf{x} \in S^2$, then we have $b_i > -C_i$.

If $\gamma_1, \gamma_2, \gamma_3 \geq 0$, then $C_i > 0$ for any $\mathbf{x} \in int(S^2)$. This implies $a_{ii} > C_i$, from which we obtain $a_{11}a_{22}a_{33} > C_1C_2C_3$. From this fact, we may show the following proposition:

Proposition 3.2 Let $\mathbf{x} \in int(S^2)$. If $\gamma_1, \gamma_2, \gamma_3 \geq 0$, then $|A| > 0$.

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \geq 0$, then immediately we get $a_{ii} > 0$ for any $\mathbf{x} \in int(S^2)$. Since $sgn(\gamma_i) = sgn(C_i)$, we have $a_{11}a_{22}a_{33} > C_1C_2C_3$. This shows that

$$|A(\mathbf{x})| > \gamma_1\gamma_2\gamma_3\eta_1(x_1)\eta_2(x_2)\eta_3(x_3) + \frac{a_{11}}{x_1}\gamma_2^2\eta_2(x_2)\frac{\partial}{\partial x_2}\eta_2(x_2)x_2$$
$$+ \frac{a_{22}}{x_2}\gamma_3^2\eta_2(x_2)\frac{\partial}{\partial x_3}\eta_3(x_3)x_3 + \frac{a_{33}}{x_3}\gamma_1^2\eta_1(x_1)\frac{\partial}{\partial x_1}\eta_1(x_1)x_1$$
$$\geq 0.$$

Hence, $|A| > 0$ for all $\mathbf{x} \in int(S^2)$. □

It is good to note that b_1, b_2, and b_3 are minimum when their respective parameters γ_2, γ_3, and γ_1 are equal to 1, and C_i is minimum when $\gamma_i = -1$, i.e.,

$$\begin{aligned}
\min a_{11} &= 1 - \eta_3(x_3)x_3 - x_2\frac{\partial}{\partial x_1}\eta_1(x_1)x_1,\\
\min a_{22} &= 1 - \eta_1(x_1)x_1 - x_3\frac{\partial}{\partial x_2}\eta_2(x_2)x_2,\\
\min a_{33} &= 1 - \eta_2(x_2)x_2 - x_1\frac{\partial}{\partial x_3}\eta_3(x_3)x_3.
\end{aligned} \tag{3.1.5}$$

To show $a_{11}, a_{22}, a_{33} > 0$ for any strictly increasing differentiable convex function $\eta_1(x)$, $\eta_2(x)$, and $\eta_3(x)$, where $\mathbf{x} \in int(S^2)$ we will need the following lemma:

Lemma 3.3 If $f(x) \in [0,1]$ is a strictly increasing differentiable convex function, then

$$f'(x) \leq \frac{1}{1-x}$$

for any x taken in (0,1).

Proof. Since $f(x)$ is convex for $x \in (0,1)$, it lies above its tangent line at every point $\big(x, f(x)\big)$, i.e.,

$$f(x) \geq f'(t)x + c, \qquad t \in (0,1),$$

where $c = f(t) - f'(t)t$. Suppose $f'(x) > \frac{1}{1-x}$ at a point $t \in (0,1)$, then for any $x \in (0,1)$ we have

$$f(x) \geq f'(t)(x-t) + f(t) > \frac{x-t}{1-t} + f(t).$$

However, this implies $f(1) > 1$ which contradicts the fact that $f(1) \leq 1$. Hence, $f'(x) \leq \frac{1}{1-x}$. □

Consequently, from Lemma **3.3** we prove the following proposition:

Proposition 3.4 Let $\mathbf{x} \in int(S^2)$, then $a_{ii} > 0$ for $i = 1,2,3$.

Proof. Let $\mathbf{x} \in int(S^2)$, then $x_1, x_2, x_3 \in (0,1)$. By Lemma **3.3**, we get $\frac{\partial}{\partial x_i}\eta_i(x_i)x_i \leq \frac{1}{1-x_i}$ for $i = 1,2,3$. Consider a_{11}, applying the inequality to (3.1.5) gives us

$$\begin{aligned}
\min a_{11} &\geq 1 - \eta_3(x_3)x_3 - \frac{x_2}{1-x_1} \\
&= 1 - \eta_3(x_3)x_3 - \frac{1-x_1-x_3}{1-x_1} \\
&= \frac{x_3[1-\eta_3(x_3)(1-x_1)]}{1-x_1} \\
&> 0.
\end{aligned}$$

Hence, $a_{11} > 0$ for any $\mathbf{x} \in int(S^2)$. The other two cases can be proven similarly. □

In addition to previous cases, a straightforward calculation shows that given $a_{ii} = b_i + C_i \geq |C_i|$ for any $i = 1,2,3$, we obtain

$$\frac{a_{11}a_{22}a_{33}}{x_1x_2x_3} \geq |\gamma_1\gamma_2\gamma_3| \left(\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\right)\left(\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\right)\left(\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\right).$$

Then, we may prove the following proposition:

Proposition 3.5 If $a_{ii} = b_i + C_i \geq |C_i|$, then $|A| > 0$.

Proof. One may consider two cases, $\gamma_1\gamma_2\gamma_3 \leq 0$ and $\gamma_1\gamma_2\gamma_3 \geq 0$. Suppose $\gamma_1\gamma_2\gamma_3 \leq 0$, then since $\eta_1(x_1)\eta_2(x_2)\eta_3(x_3) - \left(\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\right)\left(\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\right)\left(\frac{\partial}{\partial x_1}\eta_1(x_1)x_1\right) < 0$ we get

$$
\begin{aligned}
|A(\mathbf{x})| \geq -\gamma_1\gamma_2\gamma_3 \frac{\partial}{\partial x_1}\eta_1(x_1)x_1 \frac{\partial}{\partial x_2}\eta_2(x_2)x_2 \frac{\partial}{\partial x_3}\eta_3(x_3)x_3 \\
+ \frac{a_{11}}{x_1}\gamma_2^2\eta_2(x_2)\frac{\partial}{\partial x_2}\eta_2(x_2)x_2 + \frac{a_{22}}{x_2}\gamma_3^2\eta_2(x_2)\frac{\partial}{\partial x_3}\eta_3(x_3)x_3 \\
+ \frac{a_{33}}{x_3}\gamma_1^2\eta_1(x_1)\frac{\partial}{\partial x_1}\eta_1(x_1)x_1 \\
> 0.
\end{aligned}
$$

Now suppose $\gamma_1\gamma_2\gamma_3 \geq 0$, we may cancel out the first and third terms from $|A(\mathbf{x})|$ to get

$$
\begin{aligned}
|A(\mathbf{x})| = \gamma_1\gamma_2\gamma_3\eta_1(x_1)\eta_2(x_2)\eta_3(x_3) + \frac{a_{11}}{x_1}\gamma_2^2\eta_2(x_2)\frac{\partial}{\partial x_2}\eta_2(x_2)x_2 \\
+ \frac{a_{22}}{x_2}\gamma_3^2\eta_2(x_2)\frac{\partial}{\partial x_3}\eta_3(x_3)x_3 + \frac{a_{33}}{x_3}\gamma_1^2\eta_1(x_1)\frac{\partial}{\partial x_1}\eta_1(x_1)x_1 \\
> 0.
\end{aligned}
$$

Hence, $|A| > 0$ when $a_{ii} > |C_i|, i = 1,2,3$. □

Let $\bar{V}_{\xi,2}: S^1 \to S^1$ be the restriction of $V_{\xi,2}$ on the face Γ_α. We found that the Jacobi matrix of $\bar{V}_{\xi,2}$ has the following form:

$$
J_{\bar{V}_{\xi,2}} = \begin{bmatrix} a_{ii} & a_{ij} \\ -\kappa a_{ij} & a_{jj} \end{bmatrix}, \qquad i,j \in \alpha, \qquad (3.1.6)
$$

where $\kappa > 0$. Now, we are ready to prove the following main theorem:

Theorem 3.6 Let $\gamma_1, \gamma_2, \gamma_3 \in [-1,1]$, a_{ii} be defined as in (3.1.3), and C_i as in (3.1.4). Then, the mapping $V_{\xi,2}$ defined by (3.1.1) is a homeomorphism if it satisfies one of the following conditions:

i. $\gamma_1, \gamma_2, \gamma_3$ have the same sign;

ii. $a_{ii} > |C_i|$.

Proof. By Proposition 3.4, we get that the determinant of the Jacobi matrix (3.1.6) is positive, $\left|J_{\bar{V}_{\xi,2}}\right| > 0$. Thus, $V_{\xi,2}$ maps the faces of S^2 onto itself homeomorphically. We will also show that $V_{\xi,2}$ is a local homeomorphism at any interior point of S^2.

Let $\mathbf{x} \in int(S^2)$, recall that $\left|J_{V_{\xi,2}}\right| = x_1x_2x_3|A(\mathbf{x})|$. We suppose the first case. If $sgn(\gamma_1) = sgn(\gamma_2) = sgn(\gamma_3) = 1$, then we may apply Proposition **3.2** to show that $|A| > 0 \Rightarrow \left|J_{V_{\xi,2}}\right| > 0$. Otherwise if $sgn(\gamma_1) = sgn(\gamma_2) = sgn(\gamma_3) = -1$, then by Proposition **3.1** we also have $|A| > 0 \Rightarrow \left|J_{V_{\xi,2}}\right| > 0$.

Now suppose $a_{ii} > |C_i|$, then Proposition 3.5 shows that $|A| > 0$. Hence, $\left|J_{V_{\xi,2}}\right| > 0$.

Therefore, under those two conditions we have $\left|J_{V_{\xi,2}}\right| > 0$, implying that $V_{\xi,2}$ is a local homeomorphism at each point $S^2 \in int(S^2)$. This completes the proof. □

Corollary 3.7 The mapping $V_\lambda: S^2 \to S^2$ defined by

$$V_\lambda(\mathbf{x}) = \begin{cases} x_1' = x_1[1 + (2\lambda - 1)(x_1x_2 - x_3^2)], \\ x_2' = x_2[1 + (2\lambda - 1)(x_2x_3 - x_1^2)], \\ x_3' = x_3[1 + (2\lambda - 1)(x_3x_1 - x_2^2)], \end{cases}$$

where $\lambda \in [0,1]$ is a bijection.

Corollary 3.8 The mapping $V_\xi: S^2 \to S^2$ defined by

$$V_\xi(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + \eta(x_1)x_2 - \eta(x_3)x_3), \\ x_2' = x_2(1 + \eta(x_2)x_3 - \eta(x_1)x_1), \\ x_3' = x_3(1 + \eta(x_3)x_1 - \eta(x_2)x_2), \end{cases}$$

where $\eta: [0,1] \to [0,1]$ is strictly increasing is a bijection.

Both examples in previous two corollaries satisfy the first condition of Theorem 3.6. For our next example, which satisfies the second condition; the following Lemma is needed:

Lemma 3.9 Let $x \in int(S^2)$, a_{ii} be defined as in (3.1.3), and C_i as in (3.1.4). If $\frac{\partial}{\partial x}\eta(x)x \leq \frac{1}{2(1-x)}$, then $a_{ii} > |C_i|$.

Proof. From $a_{ii} = b_i + C_i$, it is clear that $a_{ii} > |C_i|$ if $C_i > 0$. Now we suppose $C_i < 0$, then $a_{ii} > |C_i|$ is true if $b_i + 2C_i > 0$. Let us consider a_{11}, one could show that

$$\begin{aligned} b_1 + 2C_1 &= 1 - \gamma_3\eta_3(x_3)x_3 + 2x_2\gamma_1\frac{\partial}{\partial x_1}\eta_1(x_1)x_1 \\ &\geq 1 - \eta_3(x_3)x_3 - \frac{x_2}{1-x_1} \\ &= \frac{x_3[1-\eta_3(x_3)(1-x_1)]}{1-x_1} \\ &> 0. \end{aligned}$$

The other two cases can be proven in the same way. □

By above lemma, the following corollary may be proven:

Corollary 3.10 The mapping $V_r: S^2 \to S^2$ defined by

$$V_r(\mathbf{x}) = \begin{cases} x_1' = x_1\left(1 + \gamma_1 x_1^{r_1}x_2 - \gamma_3 x_3^{r_3+1}\right), \\ x_2' = x_2\left(1 + \gamma_2 x_2^{r_2}x_3 - \gamma_1 x_1^{r_1+1}\right), \\ x_3' = x_3\left(1 + \gamma_3 x_3^{r_3}x_1 - \gamma_2 x_2^{r_2+1}\right), \end{cases} \tag{3.1.7}$$

where $\gamma_1, \gamma_2, \gamma_3 \in [-1,1]$ and $r_1, r_2, r_3 \geq 1$ is a bijection.

Proof. Notice that V_r is equivalent to $V_{\xi,2}$ defined by (3.1.1) for $\eta_i(x) = x^{r_i}$, $i = 1,2,3$. Here $\eta_i(x)x = x^{r_i+1}$, we may show that

$$2(1-x)\frac{\partial}{\partial x}\eta_i(x)x = 2(1-x)(r_i+1)x^{r_i} \leq 1.$$

Since $(1-x)x^{r_i}$ has a maximum of $\left(\frac{1}{r_i+1}\right)\left(\frac{r_i}{r_i+1}\right)^{r_i}$ at $x = \frac{r_i}{r_i+1}$, we get

$$2(1-x)(r_i+1)x^{r_i} \leq 2\left(\frac{r_i}{r_i+1}\right)^{r_i}.$$

Furthermore,

$$\left(\frac{r_i}{r_i+1}\right)^{r_i} = \frac{1}{\left(\frac{r_i+1}{r_i}\right)^{r_i}} \leq \frac{1}{2}$$

since $\left(\frac{r_i+1}{r_i}\right)^{r_i}$ is monotonically increasing for $r_i \geq 1$. Hence, $2(1-x)(r_i+1)x^{r_i} \leq 1$, implying $\frac{\partial}{\partial x}\eta_i(x)x \leq \frac{1}{2(1-x)}$, satisfying Lemma 3.9. Thus, our operator satisfies the second condition of Theorem 3.6, i.e. the mapping V_r is a bijection for any $\gamma_1, \gamma_2, \gamma_3 \in [-1,1]$. □

As consequence to Theorem 3.6 we get the following corollary:

Corollary 3.11 Let $V_{\xi,2}$ be defined by $(3.1.1)$ such that it satisfies the conditions given in Theorem 3.6, then for any initial point $\mathbf{x}^{(0)} \in S^2$, its negative trajectory

$$\mathbf{x}^{(0)}, V_{\xi,2}^{(-1)}\mathbf{x}^{(0)}, V_{\xi,2}^{(-2)}\mathbf{x}^{(0)}, \ldots$$

exists.

3.2 BIJECTIVE OPERATOR WHICH IS NOT f-MONOTONE

While any LV operator which is **f**-monotone was proven to be a bijection, the converse is not known. In this section, we construct an example of a bijective operator which is not **f**-monotone.

We consider a bijective operator from Corollary 3.10. We let $r_1 = r_2 = r_3$ and get

$$V_r(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + \gamma_1 x_1^r x_2 - \gamma_3 x_3^{r+1}), \\ x_2' = x_2(1 + \gamma_2 x_2^r x_3 - \gamma_1 x_1^{r+1}), \\ x_3' = x_3(1 + \gamma_3 x_3^r x_1 - \gamma_2 x_2^{r+1}), \end{cases} \tag{3.2.1}$$

where $\gamma_1, \gamma_2, \gamma_3 \in [-1,1]$ and $r \geq 1$. Following our definition of LV operator given by (2.2.1), our corresponding f_1, f_2, f_3 for above operator are

$$f_1 = \gamma_1 x_1^r x_2 - \gamma_3 x_3^{r+1},$$
$$f_2 = \gamma_2 x_2^r x_3 - \gamma_1 x_1^{r+1},$$
$$f_3 = \gamma_3 x_3^r x_1 - \gamma_2 x_2^{r+1}.$$

From the properties of scalar product, we have

$$\begin{aligned} \langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle &= \langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} \rangle - \langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{y} \rangle \\ &= \langle \mathbf{f}(\mathbf{x}), \mathbf{x} \rangle - \langle \mathbf{f}(\mathbf{y}), \mathbf{x} \rangle - \langle \mathbf{f}(\mathbf{x}), \mathbf{y} \rangle + \langle \mathbf{f}(\mathbf{y}), \mathbf{y} \rangle. \end{aligned}$$

Since $\langle \mathbf{f}(\mathbf{x}), \mathbf{x} \rangle = \langle \mathbf{f}(\mathbf{y}), \mathbf{y} \rangle = 0$, we end up with

$$\begin{aligned} \langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle &= -\langle \mathbf{f}(\mathbf{y}), \mathbf{x} \rangle - \langle \mathbf{f}(\mathbf{x}), \mathbf{y} \rangle \\ &= \gamma_1 (y_1^r - x_1^r)(x_2 y_1 - x_1 y_2) + \gamma_2 (y_3^r - x_3^r)(x_1 y_3 - x_3 y_1) \\ &\quad + \gamma_3 (y_2^r - x_2^r)(x_3 y_2 - x_2 y_3). \end{aligned}$$

By Definition 2.11, operator (3.2.1) is called **f**-monotone if $\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq 0$ for any $\mathbf{x}, \mathbf{y} \in S^2$. Thus, it is sufficient to provide counter example for some $\mathbf{x}, \mathbf{y} \in S^2$. To simplify our calculation, we let $\mathbf{x} = (q, 1-q, 0)$ and $\mathbf{y} = (1-q, 0, q)$ to get

$$\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = (1-q)^{r+1}[\gamma_1(1-q) + \gamma_3 q] + q^r[\gamma_2 q^2 - \gamma_1(1-q)^2].$$

From here, we deduce that for any $r \geq 1$, exists $q \in \left[\frac{1}{2}, 1\right]$ such that

$$\gamma_1 > \frac{\gamma_3 q(1-q)^{r+1} + \gamma_2 q^{r+2}}{[q^r - (1-q)^r](1-q)^2},$$

where

$$\gamma_3 < -\frac{q^{r+2}\gamma_2}{q(1-q)^{r+1}} + \frac{[q^r - (1-q)^r](1-q)^2}{q(1-q)^{r+1}}$$

and

$$\gamma_2 < \frac{[q - (1-q)^r](1-q)^2}{q^{r+2}}$$

satisfying $\gamma_1, \gamma_2, \gamma_3 \in (0,1]$. These conditions yield $\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle < 0$, i.e., operator (3.2.1) is not **f**-monotone. A more concrete example is given below:

Example 3.12 Consider operator (3.2.1). Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $r \geq 1$. Take $q = \frac{2}{3}$, we can choose

$$\gamma_2 < \frac{1}{4}\left(1 - \frac{1}{2^r}\right).$$

Let $\gamma_2 = \frac{1}{4}\left(1 - \frac{1}{2^r}\right) - \varepsilon_1 > 0$, we can choose

$$\gamma_3 < 2^{r+1}\varepsilon_1.$$

Suppose $\gamma_3 = 2^{r+1}\varepsilon_1 - \varepsilon_2 > 0$, then we choose

$$\gamma_1 > \frac{2^r - 1 - 2\varepsilon_2}{2^r - 1}.$$

We let $\gamma_1 = \frac{2^r-1-2\varepsilon_2}{2^r-1} + \varepsilon_3 > 0$, then for $\mathbf{x} = (k, 1-k, 0)$ and $\mathbf{y} = (1-k, 0, k)$ we obtain

$$\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = \frac{\varepsilon_3(2^{r+1} - 4^r - 1)}{3^{r+2}(2^r - 1)} < 0$$

for any $r \geq 1$. Hence, the operator is not **f**-monotone.

A simpler example can be constructed by letting $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. Under such condition, if we similarly consider $\mathbf{x} = (q, 1-q, 0)$ and $\mathbf{y} = (1-q, 0, q)$ we will get

$$\langle \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = \gamma[(1-q)^{r+1} + q^r(2q-1)].$$

Here, we observe that it is negative for $\gamma \in [-1,0)$ and $q \in \left(\frac{1}{2}, 1\right)$. Again, the operator is not **f**-monotone.

3.3 APPLICATION ON SOLUTION OF A HAMMERSTEIN INTEGRAL EQUATION

Here, we provide an application of bijectivity of a Lotka-Volterra operator to the existence and uniqueness of solution of a Hammerstein integral equation with a degenerate kernel. Note that similar application to the existence of solution to the integral equation was previously shown for a surjective operator (F. Mukhamedov et al., 2020).

Let Ω be a compact set with a Haar measure, μ. Also, we denote by $L_1(\Omega, \mu)$ and $L_\infty(\Omega, \mu)$ the spaces of all integrable and essentially bounded measurable functions on Ω, respectively. Let $K \in L_\infty(\Omega^{r+3}, \mu^{r+3}), r \geq 1$ and $\phi \in L_1(\Omega, \mu)$, we are interested in the existence and uniqueness of the solution to the integral equation

$$x(t) = \phi(t) + \int K(t, u_1, \ldots, u_{r+2}) x(u_1) \cdots x(u_{r+2})\, d\mu(u_1) \cdots d\mu(u_{r+2}), \quad (3.3.1)$$

where $t \in \Omega$ and $x(t) \in L_1(\Omega, \mu)$.

For our application, we assume the degenerate kernel, $K(t, u_1, \ldots, u_{r+2})$ having the form

$$\begin{aligned} K = {} & a_2(u_{m+2}) \prod_{j=1}^{r+1} a_1(u_j) f_{12}(t) + a_1(u_1) \prod_{j=2}^{r+2} a_3(u_j) f_{13}(t) \\ & + a_3(u_{r+2}) \prod_{j=1}^{r+1} a_2(u_j) f_{23}(t) + a_2(u_1) \prod_{j=2}^{r+2} a_1(u_j) f_{21}(t) \\ & + a_1(u_{r+2}) \prod_{j=1}^{r+1} a_3(u_j) f_{31}(t) + a_3(u_1) \prod_{j=2}^{r+2} a_2(u_j) f_{32}(t), \qquad (3.3.2) \end{aligned}$$

where $a_k, f_{ij} \in L_\infty(\Omega, \mu)$. Also, we define

$$D = \left\{ x \in L_1(\Omega, \mu) \,\middle|\, \sum_{k=1}^{3} \int_\Omega a_k(t) x(t) d\mu(t) = 1 \text{ and } 0 \leq \int_\Omega a_k(t) x(t) d\mu(t) \leq 1 \right\}.$$

In the upcoming parts, we are going to find a unique solution to (3.3.1) which belongs in D.

Recall that two functions $f, g \in L_\infty(\Omega, \mu)$ are orthogonal if

$$\int_\Omega f(t) g(t) d\mu(t) = 0.$$

Here, we denote the orthogonality of f and g by $f \perp g$. Moreover, let us denote

$$\gamma_{ij}^{(k)} = \int f_{ij}(t) a_k(t) d\mu(t) \quad \text{and} \quad g_k = \int \phi(t) a_k(t) d\mu(t),$$

then the following theorem applies:

Theorem 3.13 Let the kernel K be given as (3.3.2), and assume it satisfies the following conditions:

i. for any $i, m, l, j \in \{1,2,3\}$, we have $f_{ij} \perp_{i\neq 1} a_1$, $f_{mj} \perp_{m\neq 2} a_2$, and $f_{lj} \perp_{l\neq 3} a_3$;

ii. for any $i, j \in \{1,2,3\}, i \neq j$, we have $\gamma_{ij}^{(i)} = -\gamma_{ji}^{(j)}$ and $|\gamma_{ij}| \leq 1$;

iii. for any $k \in \{1,2,3\}$, we have $g_k \geq 0$ such that $\sum_{k=1}^{3} g_k = 1$.

Then, equation (3.3.1) has a unique solution belonging to D.

Proof. After substitution of K into (3.3.1), we obtain

$$x(t) = \phi(t) + f_{12}(t)x_1^{r+1}x_2 + f_{23}(t)x_2^{r+1}x_3 + f_{31}(t)x_3^{r+1}x_1 + f_{21}(t)x_2x_1^{r+1} + f_{13}(t)x_1x_3^{r+1} + f_{32}(t)x_3x_2^{r+1}, \tag{3.3.3}$$

where

$$x_k = \int_\Omega a_k(t)x(t)d\mu(t).$$

From (3.3.3), we observe that it is enough to get x_1, x_2, and x_3 to get $x(t)$. If we multiply it with a_k and integrate it, by the first condition we have

$$\begin{cases} x_1 = \gamma_{12}^{(1)}x_1^{r+1}x_2 + \gamma_{13}^{(1)}x_1x_3^{r+1} + g_1, \\ x_2 = \gamma_{23}^{(2)}x_2^{r+1}x_3 + \gamma_{21}^{(2)}x_2x_1^{r+1} + g_2, \\ x_3 = \gamma_{31}^{(3)}x_3^{r+1}x_1 + \gamma_{32}^{(3)}x_3x_2^{r+1} + g_3. \end{cases} \tag{3.3.4}$$

By the second condition, we let g_1, g_2, and g_3 be the subjects and get

$$\begin{cases} g_1 = x_1\left(1 + \gamma_{21}^{(2)} x_1^r x_2 - \gamma_{13}^{(1)} x_3^{r+1}\right), \\ g_2 = x_2\left(1 + \gamma_{32}^{(3)} x_2^r x_3 - \gamma_{21}^{(2)} x_1^{r+1}\right), \\ g_3 = x_3\left(1 + \gamma_{13}^{(1)} x_2^r x_3 - \gamma_{32}^{(3)} x_2^{r+1}\right). \end{cases} \quad (3.3.5)$$

Notice the similarity of the system of equations (3.3.5) with the operator (3.2.1) taken from Corollary 3.10. If we substitute $\gamma_1 = \gamma_{21}^{(2)}, \gamma_2 = \gamma_{32}^{(3)}$, and $\gamma_3 = \gamma_{13}^{(1)}$ into operator V_r defined by (3.2.1), it is clear that we will have $V_r(\mathbf{x}) = \mathbf{g}$, where $\mathbf{g} = (g_1, g_2, g_3) \in S^2$. By Theorem 3.6, we know that V_r is a bijection as shown in Corollary 3.10. Therefore, there exist a solution $\mathbf{x} = V_r^{-1}(\mathbf{x})$ which is unique. The proof is complete. □

Next, we give an example of functions which satisfy the conditions in Theorem 3.13.

Example 3.14 Let Ω be a compact set with a Haar measure, μ and $\mathfrak{B}$ be the Borel σ-algebra of Ω. Assume that $A_1, A_2, A_3 \in \mathfrak{B}$ such that $A_i \cap A_j = \emptyset$ and $\bigcup_{k=1}^3 A_k = \Omega$ with $\mu(A_k) \geq 0$.

Assume $\lambda_1, \lambda_2, \lambda_3 \in L_\infty(\Omega)_+$, we define $a_k = \lambda_k \mathbb{I}_{A_k}, k = 1,2,3$, and

$$\begin{aligned} f_{12} &= g_1 \mathbb{I}_{A_1}, & f_{13} &= g_2 \mathbb{I}_{A_1}, \\ f_{21} &= \delta_1 \mathbb{I}_{A_2}, & f_{23} &= \delta_2 \mathbb{I}_{A_2}, \\ f_{31} &= \tau_1 \mathbb{I}_{A_3}, & f_{32} &= \tau_2 \mathbb{I}_{A_3}, \end{aligned}$$

where $g_1, g_2, \delta_1, \delta_2, \tau_1, \tau_2 \in L_\infty(\Omega, \mu)$ such that

$$\int_{A_1} g_1 d\mu = -\int_{A_2} \delta_1 d\mu \ , \qquad \int_{A_2} \delta_2 d\mu = -\int_{A_3} \tau_2 d\mu, \qquad \int_{A_1} g_2 d\mu = -\int_{A_3} \tau_1 d\mu.$$

Let $\phi \geq 0$, such that $\sum_{k=1}^3 \int_{A_k} \phi \lambda_k d\mu = 1$. Then, all the conditions in Theorem 3.13 are satisfied. Consequently, the corresponding Hammerstein integral equation (3.3.1) has a unique solution.

CHAPTER FOUR

NON-ERGODIC LOTKA-VOLTERRA OPERATOR ON 2-DIMENSIONAL SIMPLEX

Studying the dynamics of a non-linear stochastic operator in general was proven to be complicated even for one defined on 2-dimensional simplex. Hence, many classes of stochastic operator were introduced and studied instead.

In this section, we are going to investigate, under some limitation; the dynamics of a class of Lotka-Volterra operator given by $(2.5.1)$. Nevertheless, the operator we consider is a generalisation of some operators said earlier. We define it on the 2-dimensional simplex (we set $m = 3$), and instead of $\xi(x): [0,1] \to [-1,1]$ we consider a strictly increasing function $\eta(x): [0,1] \to [0,1]$.

4.1 LOTKA-VOLTERRA OPERATOR $V_{\eta,2}$ DEFINED ON 2-DIMENSIONAL SIMPLEX

Consider a mapping $V_{\eta,2}: S^2 \to S^2$ defined by

$$V_{\eta,2}(\mathbf{x}) = \begin{cases} x_1' = x_1[1 + (\eta(x_1)x_2 - \eta(x_3)x_3)f(\mathbf{x})], \\ x_2' = x_2[1 + (\eta(x_2)x_3 - \eta(x_1)x_1)f(\mathbf{x})], \\ x_3' = x_3[1 + (\eta(x_3)x_1 - \eta(x_2)x_2)f(\mathbf{x})], \end{cases} \qquad (4.1.1)$$

where $\eta(x): [0,1] \to [0,1]$ is a strictly increasing function, and $f(\mathbf{x}): S^2 \to [0,1]$. Notice that $V_{\eta,2}$ is equivalent for any $\eta(x)$ and $f(\mathbf{x})$ having the same signs. Hence if we consider $\eta(x): [0,1] \to [-1,0]$ and $f(\mathbf{x}): S^2 \to [-1,0]$ instead, similar result will apply.

To get its fixed point, we let $V_{\eta,2}(\mathbf{x}) = \mathbf{x}$. Clearly, vertices $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ are the only boundary fixed points. Later, we will show that all the vertices are non-attracting. Now suppose $\mathbf{x} \in int(S^2)$, we obtain the following system of equations:

$$\begin{cases} \eta(x_1)x_2 - \eta(x_3)x_3 = 0, \\ \eta(x_2)x_3 - \eta(x_1)x_1 = 0, \\ \eta(x_3)x_1 - \eta(x_2)x_2 = 0. \end{cases} \quad (4.1.2)$$

Solution to the system of equations is the interior fixed point of $V_{\eta,2}$. We found that above equations are satisfied if and only if $x_1 = x_2 = x_3 = \frac{1}{3}$. We argue, in what follows; that if any two of x_1, x_2, and x_3 are unequal, the system of equations is not satisfied.

We suppose $x_1 > x_2$. Since η is strictly increasing we have $\eta(x_1) > \eta(x_2)$. From the left-hand side of the second and third equations, we get $\eta(x_2)x_3 - \eta(x_1)x_1 < \eta(x_1)(x_3 - x_1)$ and $\eta(x_3)x_1 - \eta(x_2)x_2 > [\eta(x_3) - \eta(x_2)]x_2$, respectively. These imply $x_3 \geq x_1$ and $\eta(x_3) \leq \eta(x_2)$, i.e., $x_1 \leq x_3 \leq x_2$. However, $x_1 \leq x_2$ contradicts $x_1 > x_2$. Similarly if $x_1 < x_2$, then $\eta(x_2)x_3 - \eta(x_1)x_1 > \eta(x_1)(x_3 - x_1)$ and $(x_3)x_1 - \eta(x_2)x_2 < [\eta(x_3) - \eta(x_2)]x_2$. We get $x_3 \leq x_1$ and $x_3 \geq x_2$, or $x_2 \leq x_3 \leq x_1$; which is contradictory to $x_1 < x_2$. Hence, since x_1 is neither greater nor less than x_2, the only possible solution is $x_1 = x_2$.

Following the fact that $x_1 = x_2$, we get $\eta(x_2)x_3 - \eta(x_1)x_1 = \eta(x_2)(x_3 - x_1) = 0$ and $\eta(x_3)x_1 - \eta(x_2)x_2 = [\eta(x_3) - \eta(x_2)](x_1 - x_2) = 0$ from the second and third equations, respectively. From here it is clear that $x_1 = x_2 = x_3$, hence the following proposition:

Proposition 4.1 The operator $V_{\eta,2}$ defined by (4.1.1) has a unique interior fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

4.2 LYAPUNOV FUNCTION OF OPERATOR $V_{\eta,2}$

Before we get to prove its Lyapunov function, we will need the following lemma:

Lemma 4.2 Let $\eta: [0,1] \to [0,1]$ be a strictly increasing function, then the inequality

$$\eta(a)b - \eta(c)c + \eta(b)c - \eta(a)a + \eta(c)a - \eta(b)b \leq 0 \qquad (4.2.1)$$

is true for any $a, b, c > 0$. Furthermore, equality holds only for $a = b = c$.

Proof. Let $a \geq b \geq c$, then $\eta(a) \geq \eta(b) \geq \eta(c)$. Using rearrangement inequality, we get

$$\eta(a)a + \eta(b)b + \eta(c)c \geq \eta(a)b + \eta(b)c + \eta(c)a.$$

Moving all the terms to the right-hand side, we get inequality (4.2.1). Furthermore, if $a > b > c$, we have $\eta(a)a + \eta(b)b + \eta(c)c > \eta(a)b + \eta(b)c + \eta(c)a$. If $a = b > c$, then we are left with

$$\eta(b)(b - c) + \eta(c)(c - b) = [\eta(b) - \eta(c)](b - c) \geq 0.$$

Obviously, if $b \neq c$, then $[\eta(b) - \eta(c)](b - c) > 0$. The same can be shown for $a > b = c$. Thus, equality occurs only for $a = b = c$. □

By above lemma, we claim that $\varphi(\mathbf{x}) = x_1 x_2 x_3$ is a decreasing Lyapunov function of $V_{\eta,2}$ for all $\mathbf{x} \in int(S^2)$. Consequently, the trajectory of $V_{\eta,2}$, for any initial point $\mathbf{x}^{(0)} \in int(S^2) \setminus \{\mathbf{c}\}$; approaches the boundary, ∂S^2 of the 2-dimensional simplex. Our finding could be summarized as follows:

Proposition 4.3 The function $\varphi(\mathbf{x}) = x_1 x_2 x_3$ is a decreasing Lyapunov function of $V_{\eta,2}$, i.e., $\varphi(V_{\eta,2}(\mathbf{x})) \leq \varphi(\mathbf{x})$ for all $\mathbf{x} \in int(S^2)$. Moreover, the limiting set $\omega(\mathbf{x}^{(0)})$ lies in ∂S^2 for any initial point $\mathbf{x}^{(0)} \in int(S^2)$.

Proof. Let $\varphi(\mathbf{x}) = x_1 x_2 x_3$, then $\varphi\left(V_{\eta,2}(\mathbf{x})\right) = \varphi(\mathbf{x})\psi(\mathbf{x})$, where

$$\psi(\mathbf{x}) = [1 + (\eta(x_1)x_2 - \eta(x_3)x_3)f(\mathbf{x})][1 + (\eta(x_2)x_3 - \eta(x_1)x_1)f(\mathbf{x})][1 + (\eta(x_3)x_1 - \eta(x_2)x_2)f(\mathbf{x})].$$

Since $f(\mathbf{x}) \geq 0$, using the inequality of arithmetic and geometric means (AM-GM inequality) together with Lemma 4.2 we obtain

$$\psi(\mathbf{x}) \leq \left[1 + \frac{\eta(x_1)x_2 - \eta(x_3)x_3 + \eta(x_2)x_3 - \eta(x_1)x_1 + \eta(x_3)x_1 - \eta(x_2)x_2}{3} f(\mathbf{x})\right]^3$$
$$\leq 1.$$

Therefore, $\varphi\big(V_{\eta,2}(\mathbf{x})\big) \leq \varphi(\mathbf{x})$, i.e., $\varphi(\mathbf{x}) = x_1x_2x_3$ is a Lyapunov function of $V_{\xi,2}$.

Consequently, for any $\mathbf{x}^{(0)} \in int(S^2) \setminus \{\mathbf{c}\}$ we have

$$0 \leq \tau \leq \varphi\left(V_{\eta,2}^{(t+1)}(\mathbf{x}^{(0)})\right) \leq \varphi\left(V_{\eta,2}^{(t)}(\mathbf{x}^{(0)})\right) \leq \cdots \leq \varphi(\mathbf{x}^{(0)}),$$

i.e., $\lim_{t\to\infty} \varphi(\mathbf{x}^{(t)}) = \tau \geq 0$. We assume $\tau > 0$, then

$$1 = \lim_{t\to\infty} \frac{\varphi\left(V_{\eta,2}^{(t+1)}(\mathbf{x}^{(0)})\right)}{\varphi\left(V_{\eta,2}^{(t)}(\mathbf{x}^{(0)})\right)} = \lim_{t\to\infty} \psi^{(t)}(\mathbf{x}^{(0)}).$$

However, $\psi(\mathbf{x})$ has the maximum of 1 only at $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. This implies $\psi(\mathbf{x}^{(t)}) < \psi(\mathbf{c})$, thus τ has to be 0, i.e., $\omega(\mathbf{x}^{(0)}) \subset \partial S^2$ for any $\mathbf{x}^{(0)} \in int(S^2)$. □

Since $\omega(\mathbf{x}^{(0)}) \subset \partial S^2$, one may question the convergence of the trajectory of $V_{\eta,2}$. In the case of convergence, it is well known that trajectory should converge to a fixed point. In our setting, if the trajectory converges it should converge to one of the fixed points $\mathbf{e}_1$, $\mathbf{e}_2$, or $\mathbf{e}_3$. However, we will show, as in next proposition; that this is not true.

Proposition 4.4 The set $\omega(\mathbf{x}^{(0)})$ is infinite for any $\mathbf{x}^{(0)} \in int(S^2) \setminus \{\mathbf{c}\}$.

Proof. Suppose $\omega\left(\mathbf{x}^{(0)}\right) \subset \partial S^2$ is singular, then it must be either $\mathbf{e}_1, \mathbf{e}_2$, or $\mathbf{e}_3$. Assume $\mathbf{x}^{(t)} \to \mathbf{e}_1$ as $t \to \infty$, then we have $x_1^{(t)} \to 1, x_2^{(t)} \to 0$, and $x_3^{(t)} \to 0$. From (4.1.1), we know that $x_1^{(t)}, x_2^{(t)}$, and $x_3^{(t)}$ should satisfy

$$\frac{x_2^{(t)}}{\eta\left(x_3^{(t)}\right)} \geq \frac{x_1^{(t)}}{\eta\left(x_2^{(t)}\right)} \geq \frac{x_3^{(t)}}{\eta\left(x_1^{(t)}\right)}$$

since x_1 is increasing, and x_2 and x_3 are decreasing. Eventually we have

$$\frac{x_1^{(t+k)}}{\eta\left(x_2^{(t+k)}\right)} \geq \frac{x_2^{(t+k)}}{\eta\left(x_3^{(t+k)}\right)} \geq \frac{x_3^{(t+k)}}{\eta\left(x_1^{(t+k)}\right)}$$

after some $k > 0$. However, this implies $x_3^{(t+k+1)} \geq x_3^{(t+k)}$ which contradicts $\lim_{t\to\infty} x_3^{(t)} = 0$, i.e., $\mathbf{x}^{(t)}$ will not converge to $\mathbf{e}_1$. Similarly, we can show that $\mathbf{x}^{(t)}$ will not converge to either $\mathbf{e}_2$ or $\mathbf{e}_3$. Hence, $\omega\left(\mathbf{x}^{(0)}\right) \subset \partial S^2 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is infinite. □

4.3 BEHAVIOUR OF THE TRAJECTORY OF $V_{\eta,2}$

Apparently, even though the trajectory of $\mathbf{x}^{(t)}$ does not converge to a vertex; we find it moves between the neighbourhood of each vertex as t tends to infinity. Moreover, the time it spends on each neighbourhood is exponentially proportional to t. To describe such behaviour, we consider the following subset of S^2:

$$G_1 = \{x_1 \geq x_2 \geq x_3\}, \quad G_2 = \{x_1 \geq x_3 \geq x_2\}, \quad G_3 = \{x_3 \geq x_1 \geq x_2\},$$
$$G_4 = \{x_3 \geq x_2 \geq x_1\}, \quad G_5 = \{x_2 \geq x_3 \geq x_1\}, \quad G_6 = \{x_2 \geq x_1 \geq x_3\}.$$

We also consider the neighbourhood of each vertex as follows: Let $\mathbb{U}_0 \subset int(S^2)$ be the neighbourhood of $\mathbf{c}$, we denote the neighbourhood of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ as $\mathbb{U}_1 = (G_1 \cup G_2) \setminus \mathbb{U}_0$, $\mathbb{U}_2 = (G_5 \cup G_6) \setminus \mathbb{U}_0$, and $\mathbb{U}_3 = (G_3 \cup G_4) \setminus \mathbb{U}_0$, respectively. Here,

$\mathbb{U}_1, \mathbb{U}_2$, and $\mathbb{U}_3$ are compact convex sets such that $\mathbb{U}_1 \cap \mathbb{U}_2 \cap \mathbb{U}_3 = \emptyset$. We denote as $\mathbb{U}$ when we refer to one of the neighbourhoods arbitrarily.

Let us define $\hookrightarrow$ such that $G_i \hookrightarrow G_j \Leftrightarrow V_{\eta,2}(G_i) \subset G_i \cup G_j$, then the following proposition describes the path taken by $\mathbf{x}^{(t)}$ as t tends to infinity:

Proposition 4.5 For any initial point $\mathbf{x}^{(0)} \in int(S^2)$, the trajectory of $V_{\eta,2}$ moves along the path

$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow G_4 \hookrightarrow G_5 \hookrightarrow G_6 \hookrightarrow G_1$$

over the long run.

Proof. By Proposition 4.3, we have $\omega\left(\mathbf{x}^{(0)}\right) \subset \partial S^2$. Eventually, we have $V_{\eta,2}^{(t)}(\mathbf{x}) \in \partial S_\varepsilon^2$ as $t \to \infty$, where $\partial S_\varepsilon^2 = \{\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\} \mid \mathrm{dist}(\mathbf{x}, \partial S^2) < \varepsilon\}$ for sufficiently small ε. We begin by choosing $\mathbf{x} \in G_1 \cap \partial S_\varepsilon^2$, we will show that $V_{\eta,2}(G_1 \cap \partial S_\varepsilon^2) \subset G_1 \cup G_2$.

Since $\mathbf{x} \in G_1 \cap \partial S_\varepsilon^2$, we have $x_1 \geq x_2 \geq \varepsilon \geq x_3$. From (4.1.1), we have $\eta(x_1)x_2 - \eta(x_3)x_3 \geq 0$ and $\eta(x_2)x_3 - \eta(x_1)x_1 \leq 0$, i.e., $x_1' \geq x_1 \geq x_2 \geq x_2'$. One could always choose ε small enough such that $x_3' \leq 2x_3 \leq 2\varepsilon < \frac{1}{3} \leq x_1 \leq x_1'$. Hence, $x_1' \geq \max(x_2', x_3')$, or $V_{\eta,2}(\mathbf{x}) \in G_1 \cup G_2$ for sufficiently small ε.

Now suppose $\mathbf{x} \in G_2 \cap \partial S_\varepsilon^2$, then $x_1 \geq x_3 \geq \varepsilon \geq x_2$. From (4.1.1), we get $\eta(x_2)x_3 - \eta(x_1)x_1 \leq 0$ and $\eta(x_3)x_1 - \eta(x_2)x_2 \geq 0$, i.e., $x_3' \geq x_3 \geq x_2 \geq x_2'$. One could always choose ε small enough such that

$$\begin{aligned}
x_2' - x_1' &= x_2[1 + (\eta(x_2)x_3 - \eta(x_1)x_1)f(\mathbf{x})] - x_1[1 + (\eta(x_1)x_2 - \eta(x_3)x_3)f(\mathbf{x})] \\
&= x_2[1 + (\eta(x_2)x_3 - 2\eta(x_1)x_1)f(\mathbf{x})] - x_1(1 - \eta(x_3)x_3 f(\mathbf{x})) \\
&\leq \varepsilon - x_1(1 - \eta(x_3)x_3 f(\mathbf{x})) \\
&\leq 0.
\end{aligned}$$

Consequently, it is either $x_1' \geq x_3' \geq x_2'$ or $x_3' \geq x_1' \geq x_2'$, i.e., $x_2' \leq \min(x_1', x_3')$. Thus, $V_{\eta,2}(\mathbf{x}) \subset G_2 \cup G_3$ for sufficiently small ε.

From previous two arguments, we showed $G_1 \hookrightarrow G_2 \hookrightarrow G_3$. The other cases, $G_3 \hookrightarrow G_4 \hookrightarrow G_5$ and $G_5 \hookrightarrow G_6 \hookrightarrow G_1$ can be proven similarly to complete the proof. □

As the trajectory of $V_{\eta,2}$ follows the path given in Proposition 4.5, it visits each neighbourhood of the vertex in the order of $\mathbb{U}_1$ to $\mathbb{U}_3$ to $\mathbb{U}_2$, and back to $\mathbb{U}_1$ again. Also, the time it spends in one neighbourhood is exponentially greater than the time it spends in the previous neighbourhood – the time spent on each neighbourhood is inversely proportional to how close the trajectory is from the boundary of S^2.

An estimation of the time spent in one neighbourhood, $\mathbb{U}$ is given in the following proposition:

Proposition 4.6 Let $\mathbf{x} \notin \mathbb{U}$, $\mathbf{x}^{(k)} \in \mathbb{U}$, for $k = 1, \dots, t$, and $\mathbf{x}^{(t+1)} \notin \mathbb{U}$, then there exist a constant $A, B > 0$ such that

$$t > A \log \frac{B}{\varphi(\mathbf{x}')}.$$

Proof. Let $\mathbb{U}$ be either $\mathbb{U}_1, \mathbb{U}_2$, or $\mathbb{U}_3$. For simplicity, we only show for $\mathbb{U}_1$ since the rest can be shown similarly. From Proposition 4.5, $\mathbf{x} \notin \mathbb{U}_1$, $\mathbf{x}' \in \mathbb{U}_1$, and $\mathbf{x}^{(t+1)} \notin \mathbb{U}_1$ imply $\mathbf{x} \in G_6$, $\mathbf{x}' \in G_1$, and $\mathbf{x}^{(t+1)} \in G_3$, respectively. Equivalently, we have

$$x_2 \geq x_1 \geq x_3, \qquad x_1' \geq x_2' \geq x_3', \qquad x_3^{(t+1)} \geq x_1^{(t+1)} \geq x_2^{(t+1)}.$$

Since $\max(x_1, x_2, x_3) \geq \frac{1}{3}$, we have $x_2 \geq \frac{1}{3}$, $x_1' \geq \frac{1}{3}$, and $x_3^{(t+1)} \geq \frac{1}{3}$. Then, from (4.1.1) we show

$$x_3^{(t+1)} = x_3^{(t)} \left[1 + \left(\eta\left(x_3^{(t)}\right)x_1^{(t)} - \eta\left(x_2^{(t)}\right)x_2^{(t)}\right) f\left(\mathbf{x}^{(t)}\right)\right]$$

$$
\begin{aligned}
&= x_3^{(t-1)}\left[1+\left(\eta\left(x_3^{(t-1)}\right)x_1^{(t-1)}-\eta\left(x_2^{(t-1)}\right)x_2^{(t-1}\right)f\left(\mathbf{x}^{(t-1)}\right)\right]\Big[1 \\
&\qquad + \left(\eta\left(x_3^{(t)}\right)x_1^{(t)}-\eta\left(x_2^{(t)}\right)x_2^{(t)}\right)f\left(\mathbf{x}^{(t)}\right)\Big] \\
&= x_3'\prod_{k=1}^{t}\left[1+\left(\eta\left(x_3^{(k)}\right)x_1^{(k)}-\eta\left(x_2^{(k)}\right)x_2^{(k)}\right)f\left(\mathbf{x}^{(k)}\right)\right] \\
&\le x_3' 2^t,
\end{aligned}
$$

or $\frac{x_3^{(t+1)}}{x_3'} \le 2^t$. Furthermore, using the Lyapunov function $\varphi(\mathbf{x}') = x_1'x_2'x_3'$ we get

$$
2^t \ge \frac{x_3^{(t+1)}}{x_3'} \ge \frac{1}{3x_3'} = \frac{x_1'x_2'}{3\varphi(\mathbf{x}')}.
$$

Since

$$
\begin{aligned}
x_2' &= x_2[1+(\eta(x_2)x_3-\eta(x_1)x_1)f(\mathbf{x})] \\
&= x_2[x_1+x_2+x_3+(\eta(x_2)x_3-\eta(x_1)x_1)f(\mathbf{x})] \\
&\ge x_2[x_2+\eta(x_1)x_1f(\mathbf{x})+x_3+(\eta(x_2)x_3-\eta(x_1)x_1)f(\mathbf{x})] \\
&\ge x_2^2 \\
&\ge \frac{1}{9},
\end{aligned}
$$

we have

$$
\frac{x_1'x_2'}{3\varphi(\mathbf{x}')} \ge \frac{1}{81\varphi(\mathbf{x}')},
$$

giving us

$$
t \ge \log_2 \frac{1}{81\varphi(\mathbf{x}')} = \frac{1}{\log 2}\log\left(\frac{1}{81\varphi(\mathbf{x}')}\right) > A\log\left(\frac{B}{\varphi(\mathbf{x}')}\right)
$$

for some constant $A, B > 0$. □

Notice that $A \log\left(\frac{B}{\varphi(\mathbf{x}')}\right)$ approaches infinity as $\varphi(\mathbf{x}')$ tends to zero. Therefore, the time a trajectory spends on a neighbourhood is approaching infinity as $t \to \infty$. The next proposition strengthens our finding, in addition to comparing the time a trajectory spends in a neighbourhood with the time it spends outside of it.

Proposition 4.7 Let $\{v_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$ be sequences of natural number such that $\mathbf{x}^{(v_i)} \notin \mathbb{U}$, $\mathbf{x}^{(v_i+k)} \in \mathbb{U}$ for $1 \leq k \leq u_i$, and $x^{(v_i+u_i+1)} \notin \mathbb{U}$. Then there exist a constant $C > 0$ such that $u_i > Cv_i$.

Proof. From Proposition 4.3 we get $\varphi(\mathbf{x}') = \psi(\mathbf{x})\varphi(\mathbf{x}) < \rho\varphi(\mathbf{x})$, where

$$\rho = \max_{\mathbf{x} \in int(S^2)\setminus\mathbb{U}_0} \psi(\mathbf{x}) < 1.$$

Note that

$$\begin{aligned}\varphi\left(\mathbf{x}^{(v_i+1)}\right) &= \psi\left(\mathbf{x}^{(v_i)}\right)\varphi\left(\mathbf{x}^{(v_i)}\right) \\ &= \psi\left(\mathbf{x}^{(v_i)}\right)\psi\left(\mathbf{x}^{(v_i-1)}\right)\varphi\left(x^{(v_i-1)}\right) \\ &= \varphi(\mathbf{x}')\prod_{k=1}^{v_i}\psi\left(\mathbf{x}^{(k)}\right) \\ &< \rho^{v_i}\varphi(\mathbf{x}').\end{aligned}$$

So, from Proposition 4.6 we have

$$u_i > A\log\left(\frac{B}{\varphi\left(\mathbf{x}^{(v_i+1)}\right)}\right) > A\log\left(\frac{B}{\rho^{v_i}\varphi(\mathbf{x}')}\right).$$

Since

$$A\log\left(\frac{B}{\rho^{v_i}\varphi(\mathbf{x}')}\right) = A\log\left(\frac{B}{\varphi(\mathbf{x}')}\right) + A\log\left(\frac{1}{\rho^{v_i}}\right) > v_i A\log\left(\frac{1}{\rho}\right) > Cv_i$$

for some $C > 0$, we conclude that $u_i > Cv_i$. □

Suppose $\mathbf{x}^{(0)} \in \{\mathbb{U} \cap \partial S_\varepsilon^2\}$. Since trajectory moves between neighbourhoods, it will return to the same neighbourhood it started from over time. For such reason, it is possible to construct a sequence

$$\left\{V_{\eta,2}^{(k)}(\mathbf{x})\right\}_{k=0}^{\infty} = \{\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(p_1)}, \mathbf{x}^{(p_1+1)}, \dots, \mathbf{x}^{(p_1+q_1)}, \mathbf{x}^{(p_1+q_1+1)}, \dots, \mathbf{x}^{(p_1+q_1+p_2)},$$
$$\mathbf{x}^{(p_1+q_1+p_2+1)}, \dots, \mathbf{x}^{(p_1+q_1+p_2+q_2)}, \dots\},$$

where

$$\left\{\mathbf{x}^{\left(\sum_{i=1}^{n-1}(p_i+q_i)+k\right)}\right\}_{k=1}^{p_n} \in \mathbb{U} \quad \text{and} \quad \left\{\mathbf{x}^{\left(\sum_{i=1}^{n-1}(p_i+q_i)+p_n+k\right)}\right\}_{k=1}^{q_n} \notin \mathbb{U}$$

for any $n \geq 2$. Moreover, due to Proposition 4.7 we have the following corollary:

Corollary 4.8 For any $n \geq 2$, there exist a constant $C > 0$ such that

$$p_n \geq C \sum_{i=1}^{n-1}(p_i + q_i) \quad \text{and} \quad q_n \geq C\left(\sum_{i=1}^{n-1}(p_i + q_i) + p_n\right). \tag{4.3.1}$$

4.4 NON-ERGODICITY OF $V_{\eta,2}$

Based on the results we have, we will show that operator $V_{\eta,2}$ defined by $(4.1.1)$ is non-ergodic.

Theorem 4.9 Let the mapping $V_{\eta,2}: S^2 \to S^2$ be an operator given by $(4.1.1)$. Then the limit

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} V_{\eta,2}^{(k)}(\mathbf{x}) \tag{4.4.1}$$

does not exist for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$.

Proof. Suppose that limit (4.4.1) exists for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$. Due to Proposition 4.3, 4.5, and 4.6, we know that $\omega(\mathbf{x}) \subset \partial S^2$, and its trajectory eventually belongs to $\mathbb{U}$, one of the neighbourhood of the vertices of S^2.

We assume $\lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} V_{\eta,2}^{(k)}(\mathbf{x}) = \mathbf{x}^*$ for any $\mathbf{x} \in int(S^2)$ such that $\mathbf{x}^* \neq \mathbf{c}$. Suppose $\mathbf{x}^* \notin \mathbb{U}$, and v_i, u_i be as in Proposition 4.7. If we denote $\delta = \mathrm{dist}(\mathbf{x}^*, \mathbb{U})$ and $\lambda_i = \frac{u_i}{v_i}$, then clearly $\delta > 0$ and $\lambda_i > C$. Let

$$\begin{aligned}
\mathbf{x}_{uv} &= \frac{1}{v_i + u_i}\sum_{k=0}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) \\
&= \frac{1}{v_i + u_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) + \frac{1}{v_i + u_i}\sum_{k=v_i}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) \\
&= \frac{1}{1+\lambda_i}\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) + \frac{\lambda_i}{1+\lambda_i}\frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x}). \qquad (4.4.2)
\end{aligned}$$

Consequently, as $i \to \infty$ we obtain

$$\begin{aligned}
0 &= \mathrm{dist}\left(\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}), \mathbf{x}_{uv}\right) \\
&= \left\|\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) - \mathbf{x}_{uv}\right\| \\
&= \left\|\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) - \left(\frac{1}{1+\lambda_i}\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) + \frac{\lambda_i}{1+\lambda_i}\frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x})\right)\right\| \\
&= \frac{\lambda_i}{1+\lambda_i}\left\|\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}) - \frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x})\right\| \\
&= \frac{\lambda_i}{1+\lambda_i}\mathrm{dist}\left(\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}), \frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x})\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{C}{1+C} \inf \operatorname{dist}\left(\frac{1}{v_i}\sum_{k=0}^{v_i-1} V_{\eta,2}^{(k)}(\mathbf{x}), \frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1} V_{\eta,2}^{(k)}(\mathbf{x})\right) \\
&\geq \frac{C}{1+C} \operatorname{dist}(\mathbf{x}^*, \mathbb{U}) \\
&= \frac{C}{1+C}\delta \\
&> 0,
\end{aligned}
$$

a contradiction. Thus, the time average $\frac{1}{t}\sum_{k=0}^{t-1} V_{\eta,2}^{(k)}(\mathbf{x})$ does not converge. □

To summarise our result, we showed that trajectory of $V_{\eta,2}$ eventually moves along the boundary following the path given in Proposition 4.5. Furthermore, as it oscillates between the neighbourhood of each vertex, the time it spends in one neighbourhood increases dramatically for every consecutive visit. Finally, Theorem 4.9 conclude that $V_{\eta,2}$ is non-ergodic. Note that we provide no further description on the property of $\omega(\mathbf{x}) \subset \partial S^2 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

As a prologue to the next chapter, we present the following corollary:

Corollary 4.10 A mapping $V_{r,2}: S^2 \to S^2$ defined by

$$
V_{r,2}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2 - x_3^{r+1}), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 + x_3^r x_1 - x_2^{r+1}), \end{cases}
$$

where $r \geq 1$ is non-ergodic.

CHAPTER FIVE

ON REGULARITY OF A FAMILY OF LOTKA-VOLTERRA STOCHASTIC OPERATOR

In this chapter, we consider a convex combination of two different classes of LV operator. The first class of LV operators belongs to $V_{\eta,2}$ which was proven to be non-ergodic in the previous chapter. The other class will be a regular operator. We then provide descriptions on the dynamics of the resultant operator.

Such kind of study is not new. Similar studies, as discussed in Chapter 2; were done by Vallander (2013), and Jamilov and Reinfelds (2021). Contrary to our study, however; those studies were done on two operators of similar order.

5.1 CONVEX COMBINATION OF TWO LOTKA-VOLTERRA OPERATORS

We consider a mapping $W_\theta: S^2 \to S^2$ defined by $W_\theta(\mathbf{x}) = \theta W_a(\mathbf{x}) + (1-\theta)W_b(\mathbf{x})$, where $\theta \in [0,1]$, and

$$W_a(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2 - x_3^{r+1}), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 + x_3^r x_1 - x_2^{r+1}), \end{cases} \quad W_b(\mathbf{x}) = \begin{cases} x_1'(1 + x_3^2 - x_1 x_2), \\ x_2'(1 + x_1^2 - x_2 x_3), \\ x_3'(1 + x_2^2 - x_3 x_1) \end{cases}$$

are two LV operators defined on 2-dimensional simplex. Note that W_a is non-ergodic as shown in Corollary 4.10, and W_b is regular. If $r = 1$, the operator is reduced to the one studied by Jamilov and Reinfelds (2021), in which regularity was proven for $\theta < \frac{1}{2}$, and $\theta > \frac{1}{2}$ implies non-ergodicity. Hence, we are interested in the case of $r > 1$.

Let us rewrite $W_\theta(\mathbf{x})$ as

$$W_\theta(\mathbf{x}) = \begin{cases} x_1' = x_1\{1 + x_1 x_2[(x_1^{r-1} + 1)\theta - 1] - x_3^2[(x_3^{r-1} + 1)\theta - 1]\}, \\ x_2' = x_2\{1 + x_2 x_3[(x_2^{r-1} + 1)\theta - 1] - x_1^2[(x_1^{r-1} + 1)\theta - 1]\}, \\ x_3' = x_3\{1 + x_3 x_1[(x_3^{r-1} + 1)\theta - 1] - x_2^2[(x_2^{r-1} + 1)\theta - 1]\}, \end{cases} \quad (5.1.1)$$

where $\theta \in [0,1]$, and $r > 1$. From (5.1.1), we observe that each face of S^2 is invariant under W_θ, i.e., W_θ is Lotka-Volterra. It is interesting to see that (5.1.1) has the form

$$W_\theta(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + \xi(x_1)x_2 - \xi(x_3)x_3), \\ x_2' = x_2(1 + \xi(x_2)x_3 - \xi(x_1)x_1), \\ x_3' = x_3(1 + \xi(x_3)x_1 - \xi(x_2)x_2), \end{cases}$$

where $\xi(x) = x[(x^{r-1} + 1)\theta - 1] \in [-1,1]$ – a special case of operator (2.5.1).

Additionally, using substitution we get that the center $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are fixed points of W_θ. For $r = 1$, it was shown that W_θ has no fixed point on the interior of the edges Γ_{12}, Γ_{23}, and Γ_{31}. Interestingly, this is not the case for $r > 1$.

Proposition 5.1 Let $r > 1$. If $\theta \in \left(\frac{1}{2}, 1\right)$, then

$$\mathbf{x}_{12} = \left(\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}, 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}, 0 \right),$$

$$\mathbf{x}_{23} = \left(0, \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}, 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \right),$$

and

$$\mathbf{x}_{31} = \left(1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}, 0, \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \right),$$

are fixed points of W_θ.

Proof. We will look for fixed points on all the edges Γ_{12}, Γ_{23}, and Γ_{31}. We consider the first case by letting $x_3 = 0$. Let $W_\theta(\mathbf{x}) = \mathbf{x}$, then we get the following system of equations:

$$x_1x_2[(x_1^{r-1}+1)\theta-1]=0,$$
$$x_1^2[(x_1^{r-1}+1)\theta-1]=0.$$

Solving the system above gives us $x_1=\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$. Consequently, from $x_1+x_2=1$ we get $x_2=1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$.

Now, suppose $x_1=0$. After we solve the resulting system of equations

$$x_2x_3[(x_2^{r-1}+1)\theta-1]=0,$$
$$x_2^2[(x_2^{r-1}+1)\theta-1]=0,$$

we yield $x_2=\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ and $x_3=1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$. Doing the same for $x_2=0$, we get $x_3=\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ and $x_1=1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ from

$$x_3x_1[(x_3^{r-1}+1)\theta-1]=0,$$
$$x_3^2[(x_3^{r-1}+1)\theta-1]=0.$$

In addition, since $x_i,x_j\in(0,1)$ for $x_i,x_j\in int(\Gamma_{ij})$, we have $\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\in(0,1)$, i.e. $\theta\in\left(\frac{1}{2},1\right)$. Hence, $\mathbf{x}_{12}=\left(\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}},1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}},0\right)$, $\mathbf{x}_{23}=\left(0,\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}},1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right)$, and $\mathbf{x}_{31}=\left(1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}},0,\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right)$ are fixed points of W_θ. □

Note that W_θ is regular for $\theta=0$, and non-ergodic for $\theta=1$. Hence, we are going to exclude $\theta=1$ from our results later. In the upcoming sections, we are going to study the stability of each fixed point, then continue to analyse the limiting behaviour of W_θ for some value of parameter θ.

5.2 STABILITY OF FIXED POINTS OF W_θ

Since W_θ maps the simplex, S^2 onto itself, it follows that $x_1 + x_2 + x_3 = 1$ and $x_1' + x_2' + x_3' = 1$. It is sufficient to study two of the elements, since the other one can be derived from them. For example, by having $\lim_{t\to\infty} x_1^{(t)} = A$ and $\lim_{t\to\infty} x_2^{(t)} = B$, it follows that $\lim_{t\to\infty} x_3^{(t)} = \lim_{t\to\infty} 1 - x_1^{(t)} - x_2^{(t)} = 1 - A - B$.

Thus, to simplify our analysis, instead of W_θ we will study the operator $\overline{W}_{\theta_{ij}} = \theta\overline{W}_i + (1-\theta)\overline{W}_j$ for $(i,j) = \{(1,2),(2,3),(3,1)\}$, where

$$\overline{W}_i(\mathbf{x}) = \begin{cases} x_i' = x_i\left(1 + x_i^r x_j - \left(1 - x_i - x_j\right)^{r+1}\right), \\ x_j' = x_j\left(1 + x_j^r\left(1 - x_i - x_j\right) - x_i^{r+1}\right), \end{cases}$$

and

$$\overline{W}_j(\mathbf{x}) = \begin{cases} x_i' = x_i\left(1 + \left(1 - x_i - x_j\right)^{r+1} - x_i x_j\right), \\ x_j' = x_j\left(1 + x_i^2 - x_j\left(1 - x_i - x_j\right)\right). \end{cases}$$

Here, $\overline{W}_{\theta_{ij}}$ maps $\overline{I_S^2}$ to itself, where $\overline{I_S^2} = \left\{\left(x_i, x_j\right) \in \mathbb{R}^2 \mid x_i, x_j \geq 0 \text{ and } x_i + x_j \leq 1\right\}$.

We notice that we can simply apply permutation to get the Jacobi matrix for each case of i,j. We will use this fact to our advantage later, to simplify the problem as we study the stability of the fixed points $\mathbf{x}_{12}, \mathbf{x}_{23}$, and $\mathbf{x}_{31}$. As for the fixed points $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, it is sufficient to consider only one of the cases.

For example, if $(i,j) = (1,2)$, then $\overline{W}_{\theta_{12}} = \theta\overline{W}_1 + (1-\theta)\overline{W}_2$ (for convenience we will denote $\overline{W}_{\theta_{12}}$ as $\overline{W}_\theta$), where

$$\overline{W}_1(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2 - (1 - x_1 - x_2)^{r+1}), \\ x_2' = x_2(1 + x_2^r(1 - x_1 - x_2) - x_1^{r+1}), \end{cases}$$

and

$$\overline{W}_2(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + (1 - x_1 - x_2)^{r+1} - x_1x_2), \\ x_2' = x_2\left(1 + x_1^2 - x_2(1 - x_1 - x_2)\right). \end{cases}$$

Then, the Jacobi matrix of $\overline{W}_1$ and $\overline{W}_2$, after replacing $1 - x_1 - x_2$ with x_3; are given by

$$J_1(\mathbf{x}) = \begin{bmatrix} 1 + (r+1)x_1^r x_2 - x_3^{r+1} + (r+1)x_1x_3^r & x_1^{r+1} + (r+1)x_1x_3^r \\ -x_2^{r+1} - (r+1)x_1^r x_2 & 1 + (r+1)x_2^r x_3 - x_2^{r+1} - x_1^{r+1} \end{bmatrix}$$

and

$$J_2(\mathbf{x}) = \begin{bmatrix} 1 + x_3^2 - 2x_1x_3 - 2x_1x_2 & -2x_1x_3 - x_1^2 \\ 2x_1x_2 + x_2^2 & 1 + x_1^2 - 2x_2x_3 + x_2^2 \end{bmatrix}.$$

Also, note that Jacobi matrix of $\overline{W}_\theta$ is

$$\begin{aligned} J_\theta = \begin{bmatrix} \frac{\partial \overline{W}_{\theta 1}}{\partial x_1} & \frac{\partial \overline{W}_{\theta 1}}{\partial x_2} \\ \frac{\partial \overline{W}_{\theta 2}}{\partial x_1} & \frac{\partial \overline{W}_{\theta 2}}{\partial x_2} \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\theta\overline{W}_{11} + (1-\theta)\overline{W}_{21}\right) & \frac{\partial}{\partial x_2}\left(\theta\overline{W}_{11} + (1-\theta)\overline{W}_{21}\right) \\ \frac{\partial}{\partial x_1}\left(\theta\overline{W}_{12} + (1-\theta)\overline{W}_{22}\right) & \frac{\partial}{\partial x_2}\left(\theta\overline{W}_{12} + (1-\theta)\overline{W}_{22}\right) \end{bmatrix} \\ &= \theta \begin{bmatrix} \frac{\partial}{\partial x_1}\overline{W}_{11} & \frac{\partial}{\partial x_2}\overline{W}_{11} \\ \frac{\partial}{\partial x_1}\overline{W}_{12} & \frac{\partial}{\partial x_2}\overline{W}_{12} \end{bmatrix} + (1-\theta) \begin{bmatrix} \frac{\partial}{\partial x_1}\overline{W}_{21} & \frac{\partial}{\partial x_2}\overline{W}_{21} \\ \frac{\partial}{\partial x_1}\overline{W}_{22} & \frac{\partial}{\partial x_2}\overline{W}_{22} \end{bmatrix} \\ &= \theta J_1 + (1-\theta)J_2. \end{aligned}$$

Proposition 5.2 The vertices $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are non-hyperbolic fixed points of operator (5.1.1).

Proof. At the vertices $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, we have

$$J_1(\mathbf{e}_1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad J_2(\mathbf{e}_1) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix};$$

$$J_1(\mathbf{e}_2) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad J_2(\mathbf{e}_2) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix};$$

and

$$J_1(\mathbf{e}_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J_2(e_3) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

From here, we get

$$J_\theta(\mathbf{e}_1) = \begin{bmatrix} 1 & 2\theta - 1 \\ 0 & 2(1-\theta) \end{bmatrix}, \qquad J_\theta(\mathbf{e}_2) = \begin{bmatrix} 1 & 0 \\ 1 - 2\theta & 2(1-\theta) \end{bmatrix},$$
$$J_\theta(\mathbf{e}_3) = \begin{bmatrix} 2(1-\theta) & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, upon finding the eigenvalues of the Jacobi matrix, we obtain a similar characteristic equation

$$|J_\theta - I\lambda| = \lambda^2 - (3 - 2\theta)\lambda + 2(1-\theta) = 0,$$

at each vertex. Its solutions are $\lambda_1 = 1$ and $\lambda_2 = 2(1-\theta)$. Hence, the vertices are non-hyperbolic fixed points of W_θ for any $\theta \in [0,1]$. □

In order to study the stability of $\mathbf{x}_{12}, \mathbf{x}_{23}$, and $\mathbf{x}_{31}$, we will consider $\overline{W}_{\theta_{31}}, \overline{W}_{\theta_{12}}$, and $\overline{W}_{\theta_{23}}$ for each case, respectively. This way, the calculation for the eigenvalues of their Jacobi matrix can be done in similar fashion. Before going any further, note that the following lemma is crucial:

Lemma 5.3 If $\theta > \frac{2^{r-1}}{2^{r-1}+1}$ (respectively $\frac{1}{2} < \theta \leq \frac{2^{r-1}}{2^{r-1}+1}$), then $\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} < 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ (resp. $\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$).

Proof. Suppose $\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} < 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then

$$\frac{1-\theta}{\theta} < \frac{1}{2^{r-1}} \Rightarrow \left(\frac{1}{2^{r-1}} + 1\right)\theta > 1 \Rightarrow \theta > \frac{2^{r-1}}{2^{r-1}+1}$$

is true. The other case can be proven likewise. □

Proposition 5.4 Let $r > 1$. If $\theta \in \left(\frac{2^{r-1}}{2^{r-1}+1}, 1\right)$, then the fixed points $\mathbf{x}_{12}, \mathbf{x}_{23}$, and $\mathbf{x}_{31}$ are saddle-nodes. Otherwise, if $\theta \in \left(\frac{1}{2}, \frac{2^{r-1}}{2^{r-1}+1}\right]$, then they are repelling.

Proof. We consider the fixed point $\mathbf{x}_{12} = \left(\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}, 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}, 0 \right)$. From $\overline{W}_{\theta_{31}} = \theta\overline{W}_3 + (1-\theta)\overline{W}_1$ we get the Jacobi matrix $J_{\theta_{31}} =$

$$\begin{bmatrix} 1 - x_2^{r+1}\theta + x_2^2(1-\theta) & 0 \\ -x_1^{r+1}\theta + x_1^2(1-\theta) & 1 + [(r+1)x_1^r x_2 - x_1^{r+1}]\theta + (x_1^2 - 2x_1x_2)(1-\theta) \end{bmatrix},$$

where $x_1 = \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ and $x_2 = 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$. Let $\left|J_{\theta_{31}} - I\lambda\right| = 0$, then we obtain the eigenvalues

$$\lambda_1 = 1 - x_2^{r+1}\theta + x_2^2(1-\theta) = 1 + x_2^2[(1-\theta) - x_2^{r-1}\theta]$$

and

$$\begin{aligned} \lambda_2 &= 1 + [(r+1)x_1^r x_2 - x_1^{r+1}]\theta + (x_1^2 - 2x_1x_2)(1-\theta) \\ &= 1 + x_1x_2[(r+1)x_1^{r-1}\theta - 2(1-\theta)] + x_1^2[(1-\theta) - x_1^{r-1}\theta]. \end{aligned}$$

After substituting $x_1 = \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ and $x_2 = 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ we get

$$\lambda_1 = 1 + \left[1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^2 \left\{(1-\theta) - \left[1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^{r-1} \theta\right\}$$

and

$$\lambda_2 = 1 + \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \left[1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right] (r-1)(1-\theta)$$

$$+\left(\frac{1-\theta}{\theta}\right)^{\frac{2}{r-1}}[(1-\theta)-(1-\theta)].$$

Since $r>1$ and $\theta\in\left(\frac{1}{2},1\right)$, it is easy to check that $0<\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}<1$, implying $\lambda_2>1$. Furthermore, according to Lemma 5.4, if $\theta>\frac{2^{r-1}}{2^{r-1}+1}$, we have

$$\lambda_1<1+\left[1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^2\left\{(1-\theta)-\left[\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^{r-1}\theta\right\}=1.$$

On the other hand, if $\frac{1}{2}<\theta\leq\frac{2^{r-1}}{2^{r-1}+1}$, then

$$\begin{aligned}\lambda_1&=1+\left[1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^2\left\{\left(\frac{1-\theta}{\theta}\right)^{\frac{r-1}{r-1}}\theta-\left[1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^{r-1}\theta\right\}\\&>1+\left[1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^2\left\{\left[1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^{r-1}\theta-\left[1-\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}\right]^{r-1}\theta\right\}\\&=1.\end{aligned}$$

Hence, if $\theta>\frac{2^{r-1}}{2^{r-1}+1}$, then $\lambda_1<1$ and $\lambda_2>1$, i.e., the fixed point $\mathbf{x}_{12}$ is a saddle-node. Otherwise, if $\frac{1}{2}<\theta\leq\frac{2^{r-1}}{2^{r-1}+1}$, then $\lambda_1>1$ and $\lambda_2>1$, i.e., the fixed point $\mathbf{x}_{12}$ is repelling.

By considering the Jacobi matrix of $\overline{W}_{\theta_{12}}$ and $\overline{W}_{\theta_{23}}$ separately for the fixed points $\mathbf{x}_{23}$ and $\mathbf{x}_{31}$, the same method can be used to complete the proof. □

5.3 LIMITING BEHAVIOUR OF W_θ UNDER SOME PARAMETER θ

Even though $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are non-hyperbolic fixed point, we find that they are attracting on S^2 for some range of parameter θ. We start by showing that the trajectory of W_θ, for any choice of $\theta \in (0,1)/\left\{\frac{1}{2}\right\}$; converges to one of the vertices for any initial point taken from the interior of the edges of S^2, excluding the fixed points.

Proposition 5.5 Let $\mathbf{x} = (x_1, x_2, x_3) \in int(\Gamma_{ij})$ and W_θ be defined as (5.1.1), then the following statements are true for $(ij) \in \{(12), (23), (31)\}$:

i. If $\theta \leq \frac{1}{2}$, then $\omega(\mathbf{x}) = \mathbf{e}_j$;

ii. if $\theta > \frac{1}{2}$ and $x_i > \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then $\omega(\mathbf{x}) = \mathbf{e}_i$ for $\mathbf{x} \neq \mathbf{x}_{ij}$;

iii. if $\theta > \frac{1}{2}$ and $x_i < \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then $\omega(\mathbf{x}) = \mathbf{e}_j$ for $\mathbf{x} \neq \mathbf{x}_{ij}$.

Proof. We choose $\mathbf{x} \in int(\Gamma_{12})$, the rest of the case can be proven in the same manner. Since $x_3 = 0$, from (5.1.1) we get

$$\begin{aligned} x_1' &= x_1\{1 + x_1 x_2[(x_1^{r-1} + 1)\theta - 1]\}, \\ x_2' &= x_2\{1 - x_1^2[(x_1^{r-1} + 1)\theta - 1]\}, \\ x_3' &= 0. \end{aligned}$$

Notice that $x_1^{r-1} + 1 < 2$ for any $r > 1$, thus $[(x_1^{r-1} + 1)\theta - 1] < 0$ for any $\theta \leq \frac{1}{2}$. Let $\theta \leq \frac{1}{2}$, then we have $x_1' < x_1$ and $x_2' > x_2$. Recall that there is no fixed point in $int(\Gamma_{12})$ for $\theta \leq \frac{1}{2}$, hence $\lim_{t\to\infty} x_1^{(t)} = 0$ and $\lim_{t\to\infty} x_2^{(t)} = 1$, i.e., $\lim_{t\to\infty} \mathbf{x}^{(t)} = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in \Gamma_{12}$.

Now suppose $\theta > \frac{1}{2}$ and $x_1 > \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, from the system of equations earlier we obtain

$$(x_1^{r-1} + 1)\theta - 1 > \left(\frac{1-\theta}{\theta} + 1\right)\theta - 1 = 0,$$

implying $x_1' > x_1$ and $x_2' < x_2$. Thus, $\lim_{t\to\infty} \mathbf{x}^{(t)} = \mathbf{e}_1$ for any $\mathbf{x}^{(0)} \in \Gamma_{12}$.

Conversely, if $\theta > \frac{1}{2}$ and $x_1 < \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then $(x_1^{r-1}+1)\theta - 1$. This implies that $x_1' < x_1$ and $x_2' > x_2$. Consequently, $\lim_{t\to\infty} \mathbf{x}^{(t)} = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in \Gamma_{12}$. □

In what follows, we will consider the parameter $\frac{3^{r-1}}{3^{r-1}+1} \leq \theta < 1$. For the sake of simplicity, we will also denote W_θ as $W_{\theta 3}$ if its parameter θ satisfies the inequality and $r > 1$. Also, it turns out that since $\frac{2^{r-1}}{2^{r-1}+1} < \frac{k^{r-1}}{k^{r-1}+1}$ for any $r > 1$ and $k > 2$, we have $\theta \geq \frac{3^{r-1}}{3^{r-1}+1} > \frac{2^{r-1}}{2^{r-1}+1}$.

We note that the interior fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is unique for $\theta \geq \frac{3^{r-1}}{3^{r-1}+1}$ in the next proposition:

Proposition 5.6 Let $\theta \geq \frac{3^{r-1}}{3^{r-1}+1}$, then the operator W_θ has a unique interior fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Proof. Let $W_\theta(\mathbf{x}) = \mathbf{x}$, $\mathbf{x} \in int(\mathrm{S}^2)$, i.e.,

$$\begin{aligned} x_1x_2[(x_1^{r-1}+1)\theta - 1] - x_3^2[(x_3^{r-1}+1)\theta - 1] = 0, \\ x_2x_3[(x_2^{r-1}+1)\theta - 1] - x_1^2[(x_1^{r-1}+1)\theta - 1] = 0, \\ x_3x_1[(x_3^{r-1}+1)\theta - 1] - x_2^2[(x_2^{r-1}+1)\theta - 1] = 0. \end{aligned}$$

From the system of equations, we observe that a solution exists if $(x_1^{r-1}+1)\theta - 1$, $(x_2^{r-1}+1)\theta - 1$, and $(x_3^{r-1}+1)\theta - 1$ have the same sign. If they are zero, then immediately we have $x_1 = x_2 = x_3$. Assuming they are positive, we get

$$x_1, x_2, x_3 > \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \Rightarrow 1 = x_1 + x_2 + x_3 > 3\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \Rightarrow \theta > \frac{3^{r-1}}{3^{r-1}+1}.$$

Otherwise, if negative; we yield

$$x_1, x_2, x_3 < \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \Rightarrow 1 = x_1 + x_2 + x_3 < 3\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \Rightarrow \theta < \frac{3^{r-1}}{3^{r-1}+1}.$$

Since we let $\theta \geq \frac{3^{r-1}}{3^{r-1}+1}$, they must be positive. However, assuming $x_1 \geq x_2 \geq x_3$ we obtain

$$x_1x_2[(x_1^{r-1}+1)\theta - 1] - x_3^2[(x_3^{r-1}+1)\theta - 1] \geq (x_1x_2 - x_3^2)[(x_3^{r-1}+1)\theta - 1]$$

from the first equation. Solution to the system of equations exists if $x_1x_2 - x_3^2 \leq 0$, since if otherwise the right-hand side of the inequality will be greater than 0.

Under our assumption, the solution is possible only if $x_1 = x_2 = x_3$. One could also check that if we have stricter inequalities $x_1 > x_2 > x_3$, $x_1 > x_2 = x_3$ or $x_1 = x_2 > x_3$, then $x_1x_2 - x_3^2 > 0$ which is not a solution. We will reach similar conclusion if we assume $x_2 \geq x_1 \geq x_3$, the rest of the cases can also be proven in the same way using the second and third equations. □

We find next that the trajectory of W_{θ_3} converges to the boundary of S^2 for any initial point $\mathbf{x}$ taken from the interior of S^2. We will show that $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing Lyapunov function for W_{θ_3}. Also, from now on the notation $\omega(\mathbf{x})$ may be used accordingly as reference to the limiting set of a trajectory of W_{θ_3} with respect to an initial point $\mathbf{x}$. Beforehand, we will need the following lemma:

Lemma 5.7 Let $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$ and $r \geq 1$, then

$$x_1^r(x_1 - x_2) + x_2^r(x_2 - x_3) + x_3^r(x_3 - x_1) \geq \frac{x_1(x_1 - x_2) + x_2(x_2 - x_3) + x_3(x_3 - x_1)}{3^{r-1}}.$$

Proof. We will show that

$$g(\mathbf{x}) = \left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1(x_1 - x_2) + \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2(x_2 - x_3)$$
$$+ \left(x_3^{r-1} - \frac{1}{3^{r-1}}\right)x_3(x_3 - x_1)$$
$$\geq 0$$

for $\mathbf{x} \in int(S^2)\backslash\{\mathbf{c}\}$.

We suppose $\min(x_1, x_2, x_3) = x_3$, i.e., either $x_1 \geq x_2 \geq x_3$ or $x_2 \geq x_1 \geq x_3$. We consider the former where $x_1 \geq x_2 \geq x_3$. Substituting $x_2 - x_3 = (x_1 - x_2) - (x_1 - x_2) + x_2 - x_3 = (x_1 - x_2) + 3x_2 - 1$ into $g(\mathbf{x})$, we get

$$g(\mathbf{x}) = \left[\left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1 + \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2\right](x_1 - x_2)$$
$$+ \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2(3x_2 - 1) + \left(x_3^{r-1} - \frac{1}{3^{r-1}}\right)x_3(x_3 - x_1).$$

Notice that $\left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)$ and $(3x_2 - 1)$ have the same sign, and since $x_3 = \min(x_1, x_2, x_3) \leq \frac{1}{3}$, we also have $\left(x_3^{r-1} - \frac{1}{3^{r-1}}\right)$ and $(x_3 - x_1)$ both negative. Hence,

$$g(\mathbf{x}) \geq \left[\left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1 + \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2\right](x_1 - x_2).$$

Since $x_1 + x_2 \geq \frac{2}{3}$, we let $x_2 = \frac{2}{3} + k - x_1, k \in \left[0, \frac{1}{3}\right)$. Consequently, from $g(\mathbf{x})$ we have

$$\left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1 + \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2$$
$$= \left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1 + \left[\left(\frac{2}{3} + k - x_1\right)^{r-1} - \frac{1}{3^{r-1}}\right]\left(\frac{2}{3} + k - x_1\right).$$

We denote the right-hand side of the equation as $g_1(x_1)$, then from

$$g_1' = rx_1^{r-1} - \frac{1}{3^{r-1}} - \left[\left(\frac{2}{3} + k - x_1\right)^{r-1} - \frac{1}{3^{r-1}}\right]$$
$$+ \left[-(r-1)\left(\frac{2}{3} + k - x_1\right)^{r-2}\right]\left(\frac{2}{3} + k - x_1\right)$$
$$= r\left[x_1^{r-1} - \left(\frac{2}{3} + k - x_1\right)^{r-1}\right],$$

we obtain a critical point $x_1 = \frac{1}{3} + \frac{k}{2}$. Moreover, from

$$g_1'' = r(r-1)\left[x_1^{r-2} + \left(\frac{2}{3} + k - x_1\right)^{r-2}\right]$$

we find that g_1 has a minimum at the critical point. Finally, it is easy to check that

$$g_1\left(\frac{1}{3} + \frac{k}{2}\right) = 2\left[\left(\frac{1}{3} + \frac{k}{2}\right)^{r-1} - \frac{1}{3^{r-1}}\right]\left(\frac{1}{3} + \frac{k}{2}\right) \geq 0.$$

This implies $\left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1 + \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2 \geq 0$ for $x_1 + x_2 \geq \frac{2}{3}$. Hence, $g(\mathbf{x}) \geq 0$.

Now consider $x_2 \geq x_1 \geq x_3$. We rewrite $g(\mathbf{x})$ as

$$g(\mathbf{x}) = \left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1(x_1 - x_2) + \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2(x_2 - x_1 + x_1 - x_3)$$
$$+ \left(x_3^{r-1} - \frac{1}{3^{r-1}}\right)x_3(x_3 - x_1)$$
$$= \left[\left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2 - \left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1\right](x_2 - x_1)$$
$$+ \left[\left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2 - \left(x_3^{r-1} - \frac{1}{3^{r-1}}\right)x_3\right](x_1 - x_3).$$

Since $x_2 = \max(x_1, x_2, x_3) \geq \frac{1}{3} \Rightarrow \left(x_2^{r-1} - \frac{1}{3^{r-1}}\right) \geq 0$, we have $\left(x_2^{r-1} - \frac{1}{3^{r-1}}\right)x_2 \geq \max\left(\left(x_1^{r-1} - \frac{1}{3^{r-1}}\right)x_1, \left(x_3^{r-1} - \frac{1}{3^{r-1}}\right)x_3\right)$. From here, it is clear that $g(\mathbf{x}) \geq 0$.

By the same technique, we could show $g(\mathbf{x}) \geq 0$ for the other two cases $\min(x_1, x_2, x_3) = x_1$ and $\min(x_1, x_2, x_3) = x_2$. □

Now, we are ready to prove the following propositions:

Proposition 5.8 The function $\varphi(\mathbf{x}) = x_1 x_2 x_3$ is a decreasing Lyapunov function of W_{θ_3} for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$.

Proof. Collecting θ from (5.1.1), we rewrite it to

$$W_\theta(\mathbf{x}) = \begin{cases} x_1' = x_1[(x_1^r + x_1)x_2 - (x_3^r + x_3)x_3]\theta + x_1(1 + x_3^2 - x_1x_2), \\ x_2' = x_2[(x_2^r + x_2)x_3 - (x_1^r + x_1)x_1]\theta + x_2(1 + x_1^2 - x_2x_3), \\ x_3' = x_3[(x_3^r + x_3)x_1 - (x_2^r + x_2)x_2]\theta + x_3(1 + x_2^2 - x_3x_1). \end{cases}$$

Let $\varphi(\mathbf{x}) = x_1 x_2 x_3$, then $\varphi\big(W_\theta(\mathbf{x})\big) = \varphi(\mathbf{x})\psi(\mathbf{x})$, where

$$\begin{aligned} \psi(\mathbf{x}) = \{1 + [(x_1^r + x_1)x_2 - (x_3^r + x_3)x_3]\theta + x_3^2 - x_1x_2\}\{1 \\ + [(x_2^r + x_2)x_3 - (x_1^r + x_1)x_1]\theta + x_1^2 - x_2x_3\}\{1 \\ + [(x_3^r + x_3)x_1 - (x_2^r + x_2)x_2]\theta + x_2^2 - x_3x_1\}. \end{aligned}$$

Then, by AM-GM inequality; we obtain $\psi(\mathbf{x}) \leq$

$$\left(1 + \frac{\begin{array}{r}[(x_1^r + x_1)(x_2 - x_1) + (x_2^r + x_2)(x_3 - x_2) + (x_3^r + x_3)(x_1 - x_3)]\theta \\ +x_1(x_1 - x_2) + x_2(x_2 - x_3) + x_3(x_3 - x_1)\end{array}}{3}\right)^3.$$

Since $\theta \geq \frac{3^{r-1}}{3^{r-1}+1}$, we apply Lemma 5.8 and get

$$\begin{aligned} \theta \geq \frac{3^{r-1}}{3^{r-1} + 1} &= \frac{1}{1 + \frac{1}{3^{r-1}}} \\ &\geq \frac{x_1(x_1 - x_2) + x_2(x_2 - x_3) + x_3(x_3 - x_1)}{(x_1^r + x_1)(x_1 - x_2) + (x_2^r + x_2)(x_2 - x_3) + (x_3^r + x_3)(x_3 - x_1)}. \end{aligned}$$

Using rearrangement inequality, we claim that $x_1^r(x_1 - x_2) + x_2^r(x_2 - x_3) + x_3^r(x_3 - x_1) \geq 0$ for any $r \geq 1$ and equality occurs only for $x_1 = x_2 = x_3$. Consequently, we have $\psi(\mathbf{x}) \leq 1$, i.e., $\varphi\left(W_{\theta_3}(\mathbf{x})\right) \leq \varphi(\mathbf{x})$. □

Proposition 5.9 The set of limit points of W_{θ_3} lies at the boundary of the S^2, i.e., $\omega(\mathbf{x}) \subset \partial S^2$ for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$.

Proof. It is clear that $\varphi(\mathbf{x}) = x_1x_2x_3 = 0$ if and only if $\mathbf{x} \in \partial S^2$. Let $\theta \geq \frac{3^{r-1}}{3^{r-1}+1}$, then by Proposition 5.8 we have $\varphi\left(W_{\theta_3}(\mathbf{x})\right) \leq \varphi(\mathbf{x})$. Then, we have a sequence

$$0 \leq \tau \leq \varphi\left(W_{\theta_3}^{(t+1)}\left(\mathbf{x}^{(0)}\right)\right) \leq \varphi\left(W_{\theta_3}^{(t)}\left(\mathbf{x}^{(0)}\right)\right) \leq \cdots \leq \varphi\left(\mathbf{x}^{(0)}\right)$$

for any $\mathbf{x}^{(0)} \in int(S^2) \setminus \{\mathbf{c}\}$.

Suppose $\tau > 0$, then

$$1 = \lim_{t\to\infty} \frac{\varphi\left(\mathbf{x}^{(t+1)}\right)}{\varphi\left(\mathbf{x}^{(t)}\right)} = \lim_{t\to\infty} \frac{\varphi\left(\mathbf{x}^{(t)}\right)\psi\left(\mathbf{x}^{(t)}\right)}{\varphi\left(\mathbf{x}^{(t)}\right)} = \lim_{t\to\infty} \psi\left(x^{(t)}\right).$$

However, $\psi\left(\mathbf{x}^{(t)}\right) = 1$ is true only at $\mathbf{x} = \mathbf{c}$, i.e., $\psi\left(\mathbf{x}^{(t)}\right) < \psi(\mathbf{c}) = 1$. Thus, τ has to be 0, i.e., $\tau = 0$ and $\omega(\mathbf{x}) \subset \partial S^2$ for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$. □

In the last chapter, we showed that for operator $V_{\eta,2}$, $\varphi(\mathbf{x}) = x_1x_2x_3$ is also a decreasing Lyapunov function for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$. We also showed that $\omega(\mathbf{x}) \subset \partial S^2 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and conclude that $V_{\eta,2}$ is non-ergodic. Now, even though the limiting set, $\omega(\mathbf{x})$ of W_{θ_3} also lies in the boundary of S^2 according to above lemma, we find that, as we will show later; $\omega(\mathbf{x}) \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$. It serves as a good example that Lyapunov function alone is insufficient in determining the asymptotic behaviour of an operator.

Proposition 5.10 Let $\mathbf{x} = (x_1, x_2, x_3) \in int(S^2) \setminus \{\mathbf{c}\}$, then the following is true for W_{θ_3}:

i. If $x_1 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then $\omega(\mathbf{x}) = \mathbf{e}_1$;

ii. If $x_2 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then $\omega(\mathbf{x}) = \mathbf{e}_2$;

iii. If $x_3 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, then $\omega(\mathbf{x}) = \mathbf{e}_3$.

Proof. Choose $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$ such that $x_1 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$. Consequently, by Lemma 5.4 we have $x_2 + x_3 \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} < 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \leq x_1$. Furthermore, from (5.1.1) we get $x_1' \geq x_1$ and $x_2' \leq x_2$ due to $x_2, x_3 \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$. Correspondingly, we have $x_1' \geq x_1 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} > \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}} \geq x_2' + x_3'$, implying $x_2', x_3' \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$. Using the same argument repeatedly, we have $x_1^{(t+1)} \geq x_1^{(t)}$ and $x_2^{(t+1)} \leq x_2^{(t)}$ for all $t \in \mathbb{N}$.

Notice that $x_1^{(t)}$ is monotone increasing and bounded by 1 while $x_2^{(t)}$ is monotone decreasing and bounded by 0, hence $\mathbf{x}^{(t)}$ converges to a fixed point as $t \to \infty$ which is, in this particular case; $\mathbf{e}_1$. Similar arguments could be made for $x_2 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ and $x_3 \geq 1 - \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$ to complete the proof. □

Let us denote $\alpha = \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{r-1}}$, we consider the following subsets of $int(S^2)$:

$$A_1 = \{\alpha \leq x_1 \leq 1 - \alpha, 0 < x_3 \leq \alpha\}, \quad A_2 = \{1 - \alpha \leq x_1 < 1, 0 < x_3 \leq \alpha\},$$
$$B_1 = \{\alpha \leq x_3 \leq 1 - \alpha, 0 < x_2 \leq \alpha\}, \quad B_2 = \{1 - \alpha \leq x_3 < 1, 0 < x_2 \leq \alpha\},$$
$$C_1 = \{\alpha \leq x_2 \leq 1 - \alpha, 0 < x_1 \leq \alpha\}, \quad C_2 = \{1 - \alpha \leq x_2 < 1, 0 < x_1 \leq \alpha\}.$$

Note that we consider the sets A_2, B_2, and C_2 in Proposition 5.10. Therefore, the limiting set of W_{θ_3}, for any initial point $\mathbf{x} \in \{A_2 \cup B_2 \cup C_2\}$ is known. Besides, it is good to know that, since $\theta \in \left[\frac{3^{r-1}}{3^{r-1}+1}, 1\right)$; we have

$$1-\alpha \geq 1-\left(\frac{\frac{3^{r-1}+1}{3^{r-1}}-1}{1}\right)^{\frac{1}{r-1}} = 1-\frac{1}{3}=\frac{2}{3}.$$

It turns out that, as we will prove in the next proposition; for any initial point $\mathbf{x} \in \{A_1 \cup B_1 \cup C_1\}$, its trajectory under W_{θ_3} will eventually belongs to $\{A_2 \cup B_2 \cup C_2\}$.

Proposition 5.11 Let $\mathbf{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in int(S^2) \setminus \{\mathbf{c}\}$, then $W_{\theta_3}^{(t)}\left(\mathbf{x}^{(0)}\right)$ converges to either $\mathbf{e}_1, \mathbf{e}_2$, or $\mathbf{e}_3$ as $t \to \infty$.

Proof. From Proposition 5.9, it is clear that $\omega\left(\mathbf{x}^{(0)}\right) \subset \partial S^2$ for any $\mathbf{x}^{(0)} \in int(S^2) \setminus \{\mathbf{c}\}$. Eventually, we have $W_{\theta_3}^{(t)}\left(\mathbf{x}^{(0)}\right) \in \partial S_\varepsilon^2$ as $t \to \infty$, where $\partial S_\varepsilon^2 = \{\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\} \mid \text{dist}(\mathbf{x}, \partial S^2) < \varepsilon\}$ for sufficiently small ε.

If exist $t \in \mathbb{N}$ such that $W_{\theta_3}^{(t)}\left(\mathbf{x}^{(0)}\right) \in \{A_2 \cup B_2 \cup C_2\}$, then Proposition 5.10 made it clear that $\omega\left(\mathbf{x}^{(0)}\right) \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Consider $\mathbf{x}^{(0)} \in \{A_1 \cup B_1 \cup C_1\} \cap \partial_\varepsilon^2$, we will show by contradiction that $\omega\left(\mathbf{x}^{(0)}\right) \not\subset \{A_1 \cup B_1 \cup C_1\}$. We suppose $\omega\left(\mathbf{x}^{(0)}\right) \subset \{A_1 \cup B_1 \cup C_1\}$, then there exist a set $\Omega \subset \{A_1 \cup B_1 \cup C_1\}$ such that $W_{\theta_3}^{(t)} \in \Omega$ as $t \to \infty$, and $W_{\theta_3}^{(t)} \notin \{A_2 \cup B_2 \cup C_2\}$ for all $t \in \mathbb{N}$.

Assume we have $t \in \mathbb{N}$ such that $\mathbf{x}^{(t)} \in A_1 \cap \partial S_\varepsilon^2$, then from (5.1.1) we have $x_1^{(t+1)} \geq x_1^{(t)}$. We claim that there exist $k \in \mathbb{N}$ such that $x_1^{(t+k)} \geq 1-\alpha \geq x_1^{(t)}$. Suppose otherwise, then $1-\alpha \geq x_1^{(t+k)} \geq x_1^{(t)}$ for all $k \in \mathbb{N}$. Since $x_1^{(t)}$ is increasing and bounded, it must converge to a number $p \leq 1-\alpha$. However, we showed previously that there is no fixed point in $int\left(\text{conv}(\mathbf{x}_{12}, \mathbf{e}_1)\right)$, the interior of the convex hull of $\mathbf{x}_{12}$ and $\mathbf{e}_1$ which is the line connecting both points on ∂S^2. Hence, convergence is not the case, and we will have some $k \in \mathbb{N}$ such that $x_1^{(t+k)} \geq 1 - \alpha \geq x_1^{(t)}$, i.e., $x_1^{(t+k)} \in A_2$.

Consequently, using the same arguments with $\mathbf{x}^{(t)} \in B_1 \cap \partial S_\varepsilon^2$ and $\mathbf{x}^{(t)} \in C_1 \cap \partial S_\varepsilon^2$ we have, for some $k \in \mathbb{N}$; $\mathbf{x}^{(t+k)} \in \{A_2 \cup B_2 \cup C_2\}$ for any $\mathbf{x}^{(t)} \in \{A_1 \cup B_1 \cup C_1\}$. Then, by Proposition 5.10 we conclude that $\omega(\mathbf{x}) \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for any $\mathbf{x} \in int(S^2) \setminus \{\mathbf{c}\}$. □

5.4 THE OPERATOR W_{θ_3} IS REGULAR

From Proposition 5.10 and 5.11, we summarise our results in the next proposition:

Proposition 5.12 Let $\mathbf{x}^{(0)} \in S^2$ and $\theta \in \left[\frac{3^{r-1}}{3^{r-1}+1}, 1\right), r > 1$, then the set of limit points of operator W_θ defined by (5.1.1) is

$$\omega\left(\mathbf{x}^{(0)}\right) = \begin{cases} \mathbf{x}^{(0)} & \text{if } \mathbf{x}^{(0)} \in Fix(W_\theta), \\ \mathbf{e}_1 & \text{if } \mathbf{x}^{(0)} \in int(\Gamma_{12}), x_1 > \alpha \text{ or } \mathbf{x}^{(\mathbf{0})} \in int(\Gamma_{31}), x_3 < \alpha, \\ \mathbf{e}_2 & \text{if } \mathbf{x}^{(0)} \in int(\Gamma_{23}), x_2 > \alpha \text{ or } \mathbf{x}^{(\mathbf{0})} \in int(\Gamma_{12}), x_1 < \alpha, \\ \mathbf{e}_3 & \text{if } \mathbf{x}^{(0)} \in int(\Gamma_{31}), x_3 > \alpha \text{ or } \mathbf{x}^{(\mathbf{0})} \in int(\Gamma_{23}), x_2 < \alpha, \\ \mathbf{e}_1, \mathbf{e}_2, \text{or } \mathbf{e}_3 & \text{if } \mathbf{x}^{(0)} \in int(S^2). \end{cases}$$

In brief, above proposition shows that the trajectory of W_θ converges for any initial point taken from the 2-dimensional simplex. Consequently, we have the following main theorem:

Theorem 5.13 Let $\theta \in \left[\frac{3^{r-1}}{3^{r-1}+1}, 1\right), r > 1$, then operator W_θ defined by (5.1.1) is regular.

Proof. Due to Proposition 5.12, if $\theta \in \left[\frac{3^{r-1}}{3^{r-1}+1}, 1\right)$, then the limit

$$\lim_{t\to\infty} W_\theta^{(t)}(\mathbf{x})$$

exists for all $\mathbf{x} \in S^2$. □

Recall previous result by Jamilov and Reinfeld, if $r = 1$ then operator (5.1.1) is non-ergodic for $\theta \in \left(\frac{1}{2}, 1\right]$. In our case, $r > 1$; we showed that operator (5.1.1) is regular for $\theta \in \left[\frac{3^{r-1}}{3^{r-1}+1}, 1\right)$. We notice that even a small change to $r = 1$ will change the property of (5.1.1) from being non-ergodic to regular for θ close enough to 1.

CHAPTER SIX

NON-ERGODIC LOTKA-VOLTERRA OPERATOR ON 3-DIMENSIONAL SIMPLEX

In Chapter 4, we showed that operator (2.5.1) is non-ergodic for $m-1=2$ if $f(\mathbf{x})$ and $\xi_i(x_i)$ have the same sign for all $i=1,2,3$. Hence, it is natural to extend such problem to a general case of $(m-1)$-dimensional simplex. However, we find that replicating our result to a higher order simplex is not straightforward due to, as we will show later; the existence of uncountable fixed points on some of its edges and the needs to construct more than one Lyapunov function to estimate the limiting set of a trajectory of the operator considered.

For S^2, it is easy to see that there are two ways of constructing a Hamiltonian cycle connecting its vertices along its 1-dimensional faces (1-faces) or edges. For a more general S^{m-1}, we have $(m-1)!$ ways of constructing a Hamiltonian cycle, or $\frac{(m-1)!}{2}$ ways if we ignore its orientation. These differences in numbers of ways that we can construct a Hamiltonian cycle give us an early insight of how our result on S^3 would differ from S^2.

From (2.5.1), it is easy to see that any $\mathbf{x}\in int(\Gamma_{ij})$ is a fixed point if $j\notin\{\pi(i),\pi^{-1}(i)\}$, where $\pi=\begin{pmatrix}1 & 2 & \dots & m-1 & m\\ 2 & 3 & \dots & m & 1\end{pmatrix}$, written in Cauchy's two line notation; is a permutation. If we exclude those points from the 1-faces, we are left with only two ways of constructing a Hamiltonian cycle which connects each vertex along $\Gamma_{12},\Gamma_{23},\dots,\Gamma_{m-1,m}$, and Γ_{m1} – either following this or the opposite order.

However, even though the eventual path taken by a non-converging trajectory of an operator that approaches the 1-faces, if exist; can be deduced from the Hamiltonian cycle, the exact path taken by such trajectory is not clear without further analysis on a simplex of order 3 or more. More caution is needed since each vertex is connected by 1 or more 1-faces consisting of fixed points, more so if they have the property of being attracting.

In this chapter, we study a class of a non-ergodic Lotka-Volterra operator, $V_{r,3}$ defined on a 3-dimensional simplex, S^3. We describe its dynamics on its 2-dimensional faces, and prove the existence of a set $X \subset int(S^3) \setminus \{\mathbf{c}\}$ such that for any $\mathbf{x} \in X$, the limit $\lim_{t\to\infty} \frac{1}{n} \sum_{k=0}^{t-1} V_{r,3}^{(k)}(\mathbf{x})$ does not exist.

6.1 LOTKA-VOLTERRA $V_{r,3}$ DEFINED ON 3-DIMENSIONAL SIMPLEX

Consider operator (2.5.1), we let $m = 4$, $\xi_i(x_i) = x_i^r, r \geq 1$, and $f(\mathbf{x}) = 1$, then we are left with a mapping $V_{r,3}: S^3 \to S^3$ defined by

$$V_{r,3}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2 - x_4^{r+1}), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 + x_3^r x_4 - x_2^{r+1}), \\ x_4' = x_4(1 + x_4^r x_1 - x_3^{r+1}). \end{cases} \quad (6.1.1)$$

Clearly, the vertices $\mathbf{e}_1 = (1,0,0,0)$, $\mathbf{e}_2 = (0,1,0,0)$, $\mathbf{e}_3 = (0,0,1,0)$, and $\mathbf{e}_4 = (0,0,0,1)$ are fixed points of the operator $V_{r,3}$.

One may check that there is no fixed point in the interior of any 2-faces of the simplex. For instance, if $x_4 = 0$, then for any $\mathbf{x} \in int(\Gamma_{123})$ we have

$$V_{r,3}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 - x_2^{r+1}), \\ x_4' = 0. \end{cases}$$

Notice that there is no solution to $V_{r,3}(\mathbf{x}) = \mathbf{x}$ from $int(\Gamma_{123})$. Similar argument applies for the other 2-faces.

As for the 1-faces or edges, there is no fixed point in the interior of $\Gamma_{12}, \Gamma_{23}, \Gamma_{34}$, and Γ_{41}. In contrast, we found that the interior of Γ_{13} and Γ_{24} consist of fixed points. We could verify this fact by letting $x_1 = x_3 = 0$, and get

$$V_{r,3}(\mathbf{x}) = \begin{cases} x_1' = 0, \\ x_2' = x_2(1+0), \\ x_3' = 0, \\ x_4' = x_4(1+0). \end{cases}$$

The same can be shown for $x_2 = x_4 = 0$, hence we get the following lemma:

Lemma 6.1 The vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$, and any point $\mathbf{x} \in int(\Gamma_{13}) \cup int(\Gamma_{24})$ are fixed points of $V_{r,3}$.

Now, let $\mathbf{x} \in int(S^3)$, then from $V_{r,3}(\mathbf{x}) = \mathbf{x}$ we obtain the following system of equations:

$$\begin{cases} x_1^r x_2 - x_4^{r+1} = 0, \\ x_2^r x_3 - x_1^{r+1} = 0, \\ x_3^r x_4 - x_2^{r+1} = 0, \\ x_4^r x_1 - x_3^{r+1} = 0. \end{cases} \qquad (6.1.2)$$

Using (6.1.2), we will prove the following proposition:

Proposition 6.2 The operator $V_{r,3}$ defined by (6.1.1) has a unique interior fixed point $\mathbf{c} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

Proof. To begin with, we suppose $x_1 > x_2$, then from the second equation of (6.1.2) we get $x_2^r x_3 - x_1^{r+1} < x_1^r(x_3 - x_1)$. This implies $x_3 \geq x_1 > x_2$, and yields $x_3^r x_4 - x_2^{r+1} > x_2^r(x_4 - x_2)$ and $x_4^r x_1 - x_3^{r+1} \leq x_3(x_4^r - x_3^r)$ from the third and fourth equation. Consequently, we have $x_4 \leq x_2$ and $x_4 \geq x_3$, but these imply $x_3 \leq x_2$, contradicting $x_3 > x_2$. Hence, solution where $x_1 > x_2$ does not exist. The same argument can be made for any pair $x_i > x_j, i, j \in \{1,2,3,4\}, i \neq j$. Hence, the only possible solution is $x_1 = x_2 = x_3 = x_4 = \frac{1}{4}$, i.e., $\mathbf{x} = \mathbf{c}$. □

6.2 STABILITY OF FIXED POINTS

In what follows, we show that the vertices and edges fixed points given in Lemma 6.1 are non-hyperbolic, while the interior fixed point as in Proposition 6.2 is a spiral source or repelling.

From (6.1.1) we have the Jacobi matrix $J_{V_{r,3}}(\mathbf{x}) =$

$$\begin{bmatrix} 1+(r+1)x_1^r x_2 - x_4^{r+1} & x_1^{r+1} & 0 & -(r+1)x_4^r x_1 \\ -(r+1)x_1^r x_2 & 1+(r+1)x_2^r x_3 - x_1^{r+1} & x_2^{r+1} & 0 \\ 0 & -(r+1)x_2^r x_3 & 1+(r+1)x_3^r x_4 - x_2^{r+1} & x_3^{r+1} \\ x_4^{r+1} & 0 & -(r+1)x_3^r x_4 & 1+(r+1)x_4^r x_1 - x_3^{r+1} \end{bmatrix}.$$

From the eigenvalues of $J_{V_{r,3}}$, we get the following propositions:

Proposition 6.3 The fixed points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_3$, and $\mathbf{x} \in int(\Gamma_{13}) \cup int(\Gamma_{24})$ are non-hyperbolic.

Proof. Choose $\mathbf{e}_1$, the rest of the vertices can be studied likewise. At $\mathbf{e}_1$, we have

$$|J_{V_{r,2}}(\mathbf{e}_1) - I\lambda| = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = \lambda(\lambda-1)^3.$$

From above we obtain $|\lambda_1| = 0$, and $|\lambda_2| = |\lambda_3| = |\lambda_4| = 1$, hence $\mathbf{e}_1$ is a non-hyperbolic fixed point.

Now suppose $x \in int(\Gamma_{13})$, then

$$\left|J_{V_{r,2}}(\mathbf{x})-I\lambda\right|=\begin{vmatrix}1-x_4^{r+1}-\lambda & 0 & 0 & 0\\ 0 & 1-\lambda & x_2^{r+1} & 0\\ 0 & 0 & 1-x_2^{r+1}-\lambda & 0\\ x_4^{r+1} & 0 & 0 & 1-\lambda\end{vmatrix}$$
$$=(1-\lambda)^2(1-x_2^{r+1}-\lambda)(1-x_4^{r+1}-\lambda),$$

from which we obtain $|\lambda_1|=|\lambda_2|=1$, and $|\lambda_3|,|\lambda_4|\in(0,1)$.

By the same calculation, we conclude that $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_3$, and $\mathbf{x}\in int(\Gamma_{13})\cup int(\Gamma_{24})$ are non-hyperbolic fixed points. □

Proposition 6.4 The fixed point $\mathbf{c}=\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ is repelling.

Proof. Let $x_4=1-x_1-x_2-x_3$ and $I_S^3=\{\mathbf{x}\in\mathbb{R}^3 \mid x_1,x_2,x_3\geq 0$ and $x_1+x_2+x_3\leq 1\}$, then we consider a mapping $\overline{V_{r,3}}\colon I_S^3\to I_S^3$ defined by

$$\overline{V_{r,3}}(\mathbf{x})=\begin{cases}x_1'=x_1(1+x_1^r x_2-(1-x_1-x_2-x_3)^{r+1}),\\ x_2'=x_2(1+x_2^r x_3-x_1^{r+1}),\\ x_3'=x_3(1+x_3^r(1-x_1-x_2-x_3)-x_2^{r+1}).\end{cases}$$

Let $J_{\overline{V}}(\mathbf{x})$ be the Jacobi matrix of $\overline{V_{r,3}}$ at a point $\mathbf{x}$, then its eigenvalues at $\mathbf{x}=\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ are λ-solutions of

$$\left|J_{\overline{V}}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)-\lambda I\right|=\begin{vmatrix}1+\frac{r}{4^{r+1}}-\lambda & \frac{1}{4^{r+1}} & 0\\ -\frac{r+1}{4^{r+1}} & 1+\frac{r}{4^{r+1}}-\lambda & \frac{1}{4^{r+1}}\\ 0 & -\frac{r+1}{4^{r+1}} & 1+\frac{r}{4^{r+1}}-\lambda\end{vmatrix}=0.$$

We denote $a=\frac{r}{4^{r+1}}$ and $b=\frac{1}{4^{r+1}}$, then

$$\left|J_{\overline{V}}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)-\lambda I\right|=\begin{vmatrix}1+a-\lambda & b & 0\\ -(a+b) & 1+a-\lambda & b\\ 0 & -(a+b) & 1+a-\lambda\end{vmatrix}$$

$$= (1 + a - \lambda)[(1 + a - \lambda)^2 + 2b(a + b)]$$
$$= 0$$

has a real solution $\lambda_1 = 1 + a > 1$, and two complex solutions

$$\lambda_2 = 1 + a + \sqrt{2b(a + b)}i, \qquad \lambda_3 = 1 + a - \sqrt{2b(a + b)}i.$$

Clearly, $|\lambda_2| = |\lambda_3| = \sqrt{(1 + a)^2 + 2b(a + b)} > 1$. Since all eigenvalues are more than 1 in absolute value, the fixed point **c** is repelling. □

6.3 LYAPUNOV FUNCTIONS OF OPERATOR $V_{r,3}$

A Lyapunov function will be proven for $V_{r,3}$, as in Chapter 3; before we begin studying its limiting behaviour. For a higher dimensional simplex such as S^3, a set of more than one Lyapunov functions will be used not only to show that a trajectory will approach ∂S^3, the boundary of S^3 but also the edges.

The following lemma will be used in our proof:

Lemma 6.5 For any $\mathbf{x} \in int(S^3)$, the inequality

$$x_1^r x_2 - x_4^{r+1} + x_2^r x_3 - x_1^{r+1} + x_3^r x_4 - x_2^{r+1} + x_4^r x_1 - x_3^{r+1} \le 0 \qquad (6.3.1)$$

is true, and equality holds only for $x_1 = x_2 = x_3 = x_4$.

Proof. Let $x_1 \ge x_2 \ge x_3 \ge x_4$, then $x_1^r \ge x_2^r \ge x_3^r \ge x_4^r$. Using rearrangement inequality we obtain

$$x_1^{r+1} + x_2^{r+1} + x_3^{r+1} + x_4^{r+1} \ge x_1^r x_2 + x_2^r x_3 + x_3^r x_4 + x_4^r x_1.$$

Moving all the terms to the right-hand side, we get (6.3.1). □

Proposition 6.6 The function $\varphi(\mathbf{x}) = x_1x_2x_3x_4$ is a decreasing Lyapunov function of $V_{r,3}$. Moreover, the limiting set $\omega(\mathbf{x}^{(0)})$ lies in ∂S^3 for any $\mathbf{x}^{(0)} \in int(S^3) \setminus \{\mathbf{c}\}$.

Proof. Let $\varphi(\mathbf{x}) = x_1x_2x_3x_4$, then $\varphi(V_{r,3}(\mathbf{x})) = \varphi(\mathbf{x})\psi(\mathbf{x})$, where

$$\psi(\mathbf{x}) = (1 + x_1^r x_2 - x_4^{r+1})(1 + x_2^r x_3 - x_1^{r+1})(1 + x_3^r x_4 - x_2^{r+1})(1 + x_4^r x_1 - x_3^{r+1}).$$

By AM-GM inequality and Lemma 6.5, we show that

$$\psi(\mathbf{x}) \le \left(1 + \frac{x_1^r x_2 - x_4^{r+1} + x_2^r x_3 - x_1^{r+1} + x_3^r x_4 - x_2^{r+1} + x_4^r x_1 - x_3^{r+1}}{4}\right) \le 1,$$

i.e., $\varphi(V_{r,3}(\mathbf{x})) \le \varphi(\mathbf{x})$.

Consequently, for any $\mathbf{x}^{(0)} \in int(S^3) \setminus \{\mathbf{c}\}$ we have

$$0 \le \tau \le \varphi(\mathbf{x}^{(t+1)}) \le \varphi(\mathbf{x}^{(t)}) \le \cdots \le \varphi(\mathbf{x}^{(0)}).$$

Suppose $\tau > 0$, then

$$1 = \lim_{t\to\infty} \frac{\varphi(\mathbf{x}^{(t+1)})}{\varphi(\mathbf{x}^{(t)})} = \lim_{t\to\infty} \psi(\mathbf{x}^{(t)}).$$

However, $\psi(\mathbf{x}) = 1$ occurs only at $\mathbf{x} = \mathbf{c}$, i.e., $\psi(\mathbf{x}^{(t)}) < \psi(\mathbf{c})$. Thus, τ must be 0, i.e., $\omega(\mathbf{x}^{(0)}) \subset \partial S^3$ for any $\mathbf{x}^{(0)} \in int(S^3) \setminus \{\mathbf{c}\}$. □

We denote as V_{ijk} for any operator $V_{r,3}$ such that $\mathbf{x} \in \Gamma_{ijk}$, where $i, j, k \in \{\overline{1,4}\}$ such that $i < j < k$. Its Lyapunov functions are given in the following proposition:

Proposition 6.7 We denote $\varphi_\gamma(\mathbf{x}) = x_i x_j x_k$, where $\gamma \notin \{i, j, k\} \subset \{\overline{1,4}\}$, then the function $\varphi_\gamma(\mathbf{x})$ is a Lyapunov function of $V_{r,3}$ for any $\mathbf{x} \in int(\Gamma_{ijk})$.

Proof. Suppose $\mathbf{x} \in int(\Gamma_{234})$, we claim that $\varphi_1(\mathbf{x}) = x_2x_3x_4$ is a Lyapunov function. Here, $x_1 = 0$. Clearly, from (6.1.1) we observe that $x_2' > x_2$ and $x_4' < x_4$. Eventually, we have $x_2^{(t)} > x_4^{(t)}$ for large enough $t \in \mathbb{N}$. Let $\varphi_1(\mathbf{x}) = x_2x_3x_4$, then $\varphi_1\left(V_{r,3}(\mathbf{x})\right) = \varphi_1(\mathbf{x})\psi_1(\mathbf{x})$, where

$$\begin{aligned}\psi_1(\mathbf{x}) &= (1 + x_2^r x_3)(1 + x_3^r x_4 - x_2^{r+1})(1 - x_3^{r+1}) \\ &\leq \left(1 + \frac{x_2^r x_3 + x_3^r x_4 - x_2^{r+1} - x_3^{r+1}}{3}\right)^3.\end{aligned}$$

Let $x_2 > x_4$, then using rearrangement inequality we obtain

$$x_2^{r+1} + x_3^{r+1} \geq x_2^r x_3 + x_3^r x_2 > x_2^r x_3 + x_3^r x_4,$$

i.e., $\psi_1(\mathbf{x}) \leq 1$. Hence, $\varphi_1\left(\mathbf{x}^{(t+1)}\right) < \varphi_1\left(\mathbf{x}^{(t)}\right)$ for any $\mathbf{x} \in int(\Gamma_{234})$ as $t \to \infty$. Consequently, we have $\lim_{t\to\infty} \varphi_i(\mathbf{x}) = 0$.

We can complete the proof using the same technique for $\varphi_2(\mathbf{x}) = x_1x_3x_4$, $\varphi_3(\mathbf{x}) = x_1x_2x_4$, and $\varphi_4(\mathbf{x}) = x_1x_2x_3$. □

Previous two propositions indicate the tendency of a trajectory of $V_{r,3}$ to approach the boundary, and eventually the edges of S^3 for any initial point $\mathbf{x} \in int(S^3) \setminus \{\mathbf{c}\}$. The following corollary is clear:

Corollary 6.8 For any $\mathbf{x} \in int(S^3)$ and $i, j \in \{\overline{1,4}\}$, we have $\omega(\mathbf{x}) \subset \bigcup_{i<j} \Gamma_{ij}$.

6.4 BEHAVIOUR OF A DIVERGING TRAJECTORY OF $V_{r,3}$.

In this section we provide the description of the path taken by a trajectory of $V_{r,3}$ for some $\mathbf{x} \in int(S^3)$. We claim that there exist a set $X \subset int(S^3)$ such that the trajectory does not converge, and the limit $\lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} V_{r,3}^{(k)}(\mathbf{x})$ does not exist for any $\mathbf{x} \in X$.

Recall that $E = \{1,2,\dots,m\}$, by $x_i \geq x_E$ we mean $x_i \geq x_j$ for any $j \in E$, i.e., $x_i \geq x_1, x_i \geq x_2, \dots, x_i \geq x_m$. We let $\beta = \{a_i\}_{i=1}^{k} \subset E$, and denote a subset of S^3 as

$$G_\beta = \{\mathbf{x} \in S^{m-1} \mid x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_k} \geq x_{E\setminus\beta}\}.$$

For example, if $m = 4$, then $G_{1234} = \{\mathbf{x} \in S^3 \mid x_1 \geq x_2 \geq x_3 \geq x_4\}$ and $G_{12} = G_{1234} \cup G_{1243}$.

Proposition 6.9 Let $\mathbf{x}^{(0)} \in int(S^3) \setminus \{\mathbf{c}\}$, then the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{e}_4$ do not belong in $\omega(\mathbf{x}^{(0)})$.

Proof. Assume $\mathbf{x}^{(t)} \to \mathbf{e}_1$ as $t \to \infty$, then $x_i^{(t)} \to 0$ for $i = 2,3,4$. From (6.1.1) it shows that $x_3 \geq x_4$, since $x_3 < x_4$ will imply $x_4' > x_4$ which contradicts $\lim\limits_{t\to\infty} x_4^{(t)} = 0$. Since $x_4' \leq x_4$, we have

$$\frac{x_4^r}{x_3} \leq \frac{x_3^r}{x_1} \quad \text{and} \quad x_3 \geq x_4.$$

From our assumption, it is clear that $\lim\limits_{t\to\infty} \frac{x_3^{(t)r}}{x_1^{(t)}} = 0$. Thus, eventually we have $N_0 \in \mathbb{N}$ such that $\frac{x_4^{(t)r}}{x_3^{(t)}} \geq \frac{x_3^{(t)r}}{x_1^{(t)}}$ for $t > N_0$. However, it implies $x_4^{t+1} \geq x_4^t$ for $t > N_0$, a contradiction to $\lim\limits_{t\to\infty} x_4^{(t)} = 0$. Thus, $\mathbf{e}_1 \notin \omega(\mathbf{x})$. The other cases can be proven similarly. □

While it is clear by Corollary 6.8 and Proposition 6.9 that $\omega(\mathbf{x}) \subset \{\bigcup_{i<j} \Gamma_{ij}\} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for $i, j \in \{\overline{1,4}\}$, it is not clear which subsets of $int(S^3)$, if exist; serve as basin of attraction to $int(\Gamma_{13})$ and $int(\Gamma_{24})$. Nevertheless, we will show the existence of a set $X \subset int(S^3)$ such that for any initial point $\mathbf{x}^{(0)} \in X$, the eventual path taken is as in the following proposition:

Proposition 6.10 There exist $X \subset int(S^3) \setminus \{\mathbf{c}\}$ such that in a long run, for any initial point $\mathbf{x}^{(0)} \in X$; the trajectory of $V_{r,3}$ moves along the path

$$G_{1243} \hookrightarrow G_{14} \hookrightarrow G_{4132} \hookrightarrow G_{43} \hookrightarrow G_{3421} \hookrightarrow G_{32} \hookrightarrow G_{2314} \hookrightarrow G_{21} \hookrightarrow G_{1243}.$$

Proof. Let us denote $\Gamma_\varepsilon = \bigcup_{i<j} \Gamma_{ij_\varepsilon}$ for $i,j \in \{\overline{1,4}\}$, where $\Gamma_{ij_\varepsilon} = \{\mathbf{x} \in int(S^3) \setminus \{\mathbf{c}\} \mid \operatorname{dist}(\mathbf{x}, \Gamma_{ij}) < \varepsilon\}$. By Proposition 6.6 and 6.7, consequently we have $\mathbf{x}^{(t)} \in \Gamma_\varepsilon$ as $t \to \infty$ where $\varepsilon > 0$ is sufficiently small. We choose $\mathbf{x} \in G_{1243} \cap \Gamma_\varepsilon$, we will show that $G_{1243} \hookrightarrow G_{14} \hookrightarrow G_{4132}$.

Let $\mathbf{x} \in G_{1243} \cap \Gamma_\varepsilon$, then $x_1' \geq x_1, x_2' \leq x_2, x_3' \leq x_3$, and $x_4' \geq x_4$, i.e., $x_1' \geq \max(x_2', x_3')$. One could always choose $x_4 = \varepsilon$ small enough such that $x_4' \leq 2x_4 = 2\varepsilon < \frac{1}{4}$, i.e., $x_1' \geq x_4'$. Hence, $V_{r,3}(G_{1243} \cap \Gamma_\varepsilon) \subset G_{1243} \cup G_{14}$.

Now suppose $\mathbf{x} \in G_{1423} \cap \Gamma_\varepsilon$, then $x_2' \leq x_2$ and $x_4' \geq x_4$. Notice that near the vertices $\mathbf{e}_1$, we have $|x_1' - x_1|, |x_3' - x_3|$, and $|x_4' - x_4|$ almost zero, while $|x_2' - x_2|$ is almost x_2. Using this argument and monotonicity of $x_2^{(t)}$, there exist $N_0 \in \mathbb{N}$ such that for any $\mathbf{x}^{(0)} \in G_{1423} \cap \Gamma_\varepsilon$ near $\mathbf{e}_1$ we have $\mathbf{x}^{(N_0+1)} \in G_{1432}$.

Next, if $\mathbf{x} \in G_{1432} \cap \Gamma_\varepsilon$, then $x_2' \leq x_2, x_3' \geq x_3$, and $x_4' \geq x_4$. We see that $|x_4' - x_4| \geq |x_3' - x_3|$ near Γ_{23}, implying $x_4' \geq x_3' \geq x_2'$. Besides, due to $x_1 > \frac{1}{4}$, one can choose $x_3 = \varepsilon$ small enough such that $x_3' \leq 2x_3 = 2\varepsilon < \frac{1}{8} \leq \frac{x_1}{2} \leq x_1(1 - x_4^{r+1}) \leq x_1'$. Consequently, we have $x_3' \leq \min(x_1', x_4')$. Thus, $V_{r,3}(G_{1432} \cap \Gamma_\varepsilon) \subset G_{1432} \cup G_{4132}$.

Based on these arguments, for sufficiently small $\varepsilon > 0$ we can show that any $\mathbf{x} \in G_{1243} \cap \Gamma_\varepsilon$ will take the path $G_{1243} \hookrightarrow G_{14} \hookrightarrow G_{4132}$. We can use the same method for the other three cases to complete the proof. □

Remark 6.11 For above proposition, we only consider $\Gamma_\varepsilon \subset X$ for $(i,j) \notin \{(1,3),(2,4)\}$. From the proof, it shows that $\varepsilon > 0$, and this implies that Γ_ε, and correspondingly X has a positive measure.

Using our notation G_β, we denote G_1, G_2, G_3, and G_4 as neighbourhood of the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{e}_4$, respectively. From Proposition 6.10, it is clear that any trajectory following the path given moves along G_1 to G_4 to G_3 to G_2 and back to G_1.

By G, we mean one of the neighbourhood arbitrarily, then the following proposition estimates the time a trajectory of $V_{r,3}$ is going to spend in each neighbourhood:

Proposition 6.12 Let $\mathbf{x}^{(0)} \notin G$, $\mathbf{x}^{(k)} \in G$ for $k = 1,2, \dots, t$, and $\mathbf{x}^{(t+1)} \notin G$, then there exist a constant $A, B > 0$ such that $t > A \cdot \log\left(\frac{B}{\varphi_{max}(\mathbf{x}')}\right)$, where $\varphi_{max}(\mathbf{x}) = \max(\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \varphi_3(\mathbf{x}), \varphi_4(\mathbf{x}))$.

Proof. Here G is either G_1, G_2, G_3, or G_4, we only provide the proof for G_4, since the proof for the rest of the neighbourhoods can be carried out in the same manner.

Consider G_4, then by Proposition 6.10 we have $\mathbf{x}^{(0)} \in G_1$, $\mathbf{x}^{(k)} \in G_4$ for $k = 1,2, \dots, t$, and $\mathbf{x}^{(t+1)} \in G_3$. From (6.1.1), for any $\mathbf{x} \in int(S^3) \setminus \{\mathbf{c}\}$ we have

$$\begin{aligned} x_3^{(t+1)} &= x_3^{(t)}\left(1 + x_3^{(t)r} x_1^{(t)} - x_2^{(t)r+1}\right) \\ &< 2x_3^{t-1}\left(1 + x_3^{(t-1)r} x_1^{(t-1)} - x_2^{(t-1)r+1}\right) \\ &< 2^t x_3'. \end{aligned}$$

Then, since $x_3^{(t+1)} \geq \frac{1}{4}$, we get

$$2^t > \frac{x_3^{(t+1)}}{x_3'} \geq \frac{1}{4x_3'} = \frac{x_1' x_4'}{4\varphi_2(\mathbf{x}')} \geq \frac{x_1' x_4'}{4\varphi_{max}(\mathbf{x}')}.$$

From the inequality, we obtain

$$2^t \varphi_{max}(\mathbf{x}') > \frac{x_1' x_4'}{4} = \frac{x_4' x_1 (1 + x_1^r x_2 - x_3^{r+1})}{4}$$

$$= \frac{x_4' x_1 [x_1 + x_2(1 + x_1^r) + x_3 + x_4(1 - x_4^r)]}{4}$$
$$\geq \frac{x_3' x_1^2}{4}$$
$$\geq \frac{1}{4^4}.$$

Thus, $2^t > \frac{1}{256\varphi_{max}(\mathbf{x}')}$ which implies

$$t > \log_2 \left(\frac{1}{256\varphi_{max}(\mathbf{x}')} \right)$$
$$= \frac{1}{\log 2} \log \left(\frac{1}{256\varphi_{max}(\mathbf{x}')} \right)$$
$$> A \cdot \log \left(\frac{B}{\varphi_{max}(\mathbf{x}')} \right),$$

for some constant $A, B > 0$. Note that $\varphi_{max}(\mathbf{x}) \to 0$ as $t \to \infty$. □

Notice that Proposition 6.12 is similar to Proposition 4.6 in Section 4.3. Using the same argument as in Chapter 4, replacing $\mathbb{U}$ with G; the upcoming proposition and corollary can be proven.

Proposition 6.13 Let $\{v_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$ be sequences of natural number such that $\mathbf{x}^{(v_i)} \notin G$, $\mathbf{x}^{(v_i+k)} \in G$ for $1 \leq k \leq u_i$, and $x^{(v_i+u_i+1)} \notin G$. Then there exist a constant $C > 0$ such that $u_i > Cv_i$.

Proof. By Proposition 6.6, we have $\varphi(\mathbf{x}') = \psi(\mathbf{x})\varphi(\mathbf{x}) < \rho\varphi(\mathbf{x})$, where

$$\rho = \max_{\mathbf{x} \in int(S^3) \setminus \{\mathbf{c}\}} \psi(\mathbf{x}) < 1.$$

Then, Proposition 6.12 implies

$$u_i > A \cdot \log \left(\frac{B}{\varphi_{max}(\mathbf{x}^{(v_i+1)})} \right)$$

$$> A \cdot \log\left(\frac{B}{\rho^{v_i}\varphi_{max}(\mathbf{x}')}\right)$$
$$= A \cdot \log\left(\frac{B}{\varphi_{max}(\mathbf{x}')}\right) + v_i A \cdot \log\left(\frac{1}{\rho}\right).$$

Since $\varphi_{max}(\mathbf{x}') \to 0$ as $v_i \to \infty$, we have

$$u_i > v_i A \cdot \log\left(\frac{1}{\rho}\right) > C v_i$$

for some constant $C > 0$. □

Suppose $\mathbf{x}^{(0)} \in G \cap \Gamma_\varepsilon$, where Γ_ε is defined as in the proof of Proposition 6.10 for a sufficiently small ε we can construct the sequence of its trajectory as follows:

$$\left\{V_{r,3}^{(k)}(\mathbf{x})\right\}_{k=0}^{\infty} = \{\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(p_1)}, \mathbf{x}^{(p_1+1)}, \dots, \mathbf{x}^{(p_1+q_1)}, \mathbf{x}^{(p_1+q_1+1)}, \dots, \mathbf{x}^{(p_1+q_1+p_2)},$$
$$\mathbf{x}^{(p_1+q_1+p_2+1)}, \dots, \mathbf{x}^{(p_1+q_1+p_2+q_2)}, \dots\},$$

such that

$$\left\{\mathbf{x}^{\left(\sum_{i=1}^{n-1}(p_i+q_i)+k\right)}\right\}_{k=1}^{p_n} \in G \quad \text{and} \quad \left\{\mathbf{x}^{\left(\sum_{i=1}^{n-1}(p_i+q_i)+p_n+k\right)}\right\}_{k=1}^{q_n} \notin G$$

for any $n \geq 2$. By previous proposition, the following corollary is apparent:

Corollary 6.14 For any $n \geq 2$, there exist a constant $C > 0$ such that

$$p_n \geq C \sum_{i=1}^{n-1}(p_i + q_i) \quad \text{and} \quad q_n \geq C\left(\sum_{i=1}^{n-1}(p_i + q_i) + p_n\right).$$

Consequently, we have the following theorem:

Theorem 6.15 Let the mapping $V_{r,3}: S^3 \to S^3$ be an operator defined by (6.1.1), then there exist $X \subset int(S^3)$ such that the limit

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}V_{r,3}^{(k)}(\mathbf{x}) \qquad (6.4.1)$$

does not exist for any $\mathbf{x}\in X$.

Proof. Suppose a limit exists for any $\mathbf{x}\in X$. Eventually, for any $\mathbf{x}\in X$ its trajectory will be in $G\cap\Gamma_\varepsilon$, one of the neighbourhoods of the vertices of S^3.

Let us assume that $\lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}V_{r,3}^{(k)}(\mathbf{x})=\mathbf{x}^*,\mathbf{x}^*\neq\mathbf{c}$ for any $\mathbf{x}\in X$. Suppose $\mathbf{x}\notin G$, and v_i,u_i be as given in Proposition 6.13. Let $\delta=dist(\mathbf{x}^*,G)$ and $\lambda_i=\frac{u_i}{v_i}$, then $\delta>0$ and $\lambda_i>C$. We also denote

$$\mathbf{x}_{uv}=\frac{1}{v_i+u_i}\sum_{k=0}^{v_1+u_i-1}V_{r,3}^{(k)}(\mathbf{x})=\frac{1}{1+\lambda_i}\frac{1}{v_i}\sum_{k=0}^{v_i-1}V_{r,3}^{(k)}(\mathbf{x})+\frac{\lambda_i}{1+\lambda_i}\frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1}V_{r,3}^{(k)}(\mathbf{x}).$$

Consequently, as $i\to\infty$ we obtain

$$\begin{aligned}
0&=\mathrm{dist}\left(\frac{1}{v_i}\sum_{k=0}^{v_i-1}V_{r,3}^{(k)}(\mathbf{x}),\mathbf{x}_{uv}\right)\\
&=\left\|\frac{1}{v_i}\sum_{k=0}^{v_i-1}V_{r,3}^{(k)}(\mathbf{x})-\left(\frac{1}{1+\lambda_i}\frac{1}{v_i}\sum_{k=0}^{v_i-1}V_{r,3}^{(k)}(\mathbf{x})+\frac{\lambda_i}{1+\lambda_i}\frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1}V_{r,3}^{(k)}(\mathbf{x})\right)\right\|\\
&=\frac{\lambda_i}{1+\lambda_i}\left\|\frac{1}{v_i}\sum_{k=0}^{v_i-1}V_{r,3}^{(k)}(\mathbf{x})-\frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1}V_{r,3}^{(k)}(\mathbf{x})\right\|\\
&\geq\frac{C}{1+C}\inf\mathrm{dist}\left(\frac{1}{v_i}\sum_{k=0}^{v_i-1}V_{r,3}^{(k)}(\mathbf{x}),\frac{1}{u_i}\sum_{k=v_i}^{v_i+u_i-1}V_{r,3}^{(k)}(\mathbf{x})\right)\\
&\geq\frac{C}{1+C}\mathrm{dist}(\mathbf{x}^*,G)\\
&=\frac{C}{1+C}\delta
\end{aligned}$$

which is greater than zero, a contradiction. Hence, the limit (6.4.1) does not exist. □

Consequently, due to Theorem 6.15 the operator given by (6.1.1) failed the ergodic hypothesis, i.e., the operator is non-ergodic.

Before we end this chapter, we investigate the dynamics on the 2-faces of S^3 in the next section.

6.5 DYNAMICS ON THE 2-DIMENSIONAL FACES OF S^3

We consider the following four cases of $V_{r,3}$ defined on the 2-faces of S^3:

$$V_{234}(\mathbf{x}) = \begin{cases} x_1' = 0, \\ x_2' = x_2(1 + x_2^r x_3), \\ x_3' = x_3(1 + x_3^r x_4 - x_2^{r+1}), \\ x_4' = x_4(1 - x_3^{r+1}), \end{cases} \quad V_{134}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 - x_4^{r+1}), \\ x_2' = 0, \\ x_3' = x_3(1 + x_3^r x_4), \\ x_4' = x_4(1 + x_4^r x_1 - x_3^{r+1}), \end{cases}$$

$$V_{124}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2 - x_4^{r+1}), \\ x_2' = x_2(1 - x_1^{r+1}), \\ x_3' = 0, \\ x_4' = x_4(1 + x_4^r x_1), \end{cases} \quad V_{123}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 - x_2^{r+1}), \\ x_4' = 0. \end{cases}$$

Note that there are only four 2-faces on S^3, each represented as above.

Let $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ be a permutation, one can show that each of above operator is a permutation of the other. Take V_{234} for example, we have $\pi_1\big(V_{234}(\pi_1(\mathbf{x}))\big) = V_{134}(\mathbf{x})$. Hence, it is sufficient for us to investigate one of the 2-faces since the rest can be investigated in the same manner.

Additionally, using the permutation $\pi_2 = \pi_1^{(2)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ we find that those operators can be grouped into two non-conjugate classes $K_1 = \{V_{234}, V_{124}\}$ and $K_2 = \{V_{134}, V_{123}\}$. We say that two operators U_1 and U_2 are conjugate, denoted by

$U_1 \sim^{\pi} U_2$; if there exist a permutation π such that $\pi^{-1}\left(U_1(\pi(\mathbf{x}))\right) = U_2$. Besides, it is easy to see that

$$\begin{aligned}\pi_2^{-1}\left(V_{234}(\pi_2(\mathbf{x}))\right) &= \pi_2^{-1}\left(0, x_4(1 + x_4^r x_1), x_1(1 + x_1^r x_2 - x_4^{r+1}), x_2(1 - x_1^{r+1})\right)\\ &= \left(x_1(1 + x_1^r x_2 - x_4^{r+1}), x_2(1 - x_1^{r+1}), 0, x_4(1 + x_4^r x_1)\right)\\ &= V_{124}(\mathbf{x}),\end{aligned}$$

and

$$\begin{aligned}\pi_2^{-1}\left(V_{134}(\pi(\mathbf{x}))\right) &= \pi_2^{-1}\left(x_3(1 - x_2^{r+1}), 0, x_1(1 + x_1^r x_2), x_2(1 + x_2^r x_3 - x_1^{r+1})\right)\\ &= (x_1(1 + x_1^r x_2), x_2(1 + x_2^r x_3 - x_1^{r+1}), x_3(1 - x_2^{r+1}), 0)\\ &= V_{123}(\mathbf{x}).\end{aligned}$$

We notice that each operator $V_{234}, V_{134}, V_{124}$, and V_{123} has two monotonous elements. Since 2-faces of S^3 are bounded, convergence of those elements is implied. Consequently, trajectories of those operators also converge. However, we find later that instead of a vertex, the trajectory converges to $int(\Gamma_{13})$ for V_{134} and V_{123}, or $int(\Gamma_{24})$ for V_{234} and V_{124}.

Let $\overline{V_{ijk}}: S^2 \to S^2$ be a restriction of $V_{r,3}$ on Γ_{ijk}, then the following proposition holds:

Proposition 6.16 The operator $\overline{V_{234}}, \overline{V_{134}}, \overline{V_{124}}$, and $\overline{V_{123}}$ are regular.

Proof. Let us consider the first operator,

$$\overline{V_{234}}(\mathbf{x}) = \begin{cases} x_2' = x_2(1 + x_2^r x_3), \\ x_3' = x_3(1 + x_3^r x_4 - x_2^{r+1}), \\ x_4' = x_4(1 - x_3^{r+1}), \end{cases} \tag{6.5.1}$$

where $\mathbf{x} = (x_2, x_3, x_4) \in S^2$. Observe from (6.5.1) that $x_2^{(t)}$ and $x_4^{(t)}$ are monotone increasing and decreasing, respectively for any $t \in \mathbb{N}$. Since x_2 is bounded above by 1, and x_4 is bounded below by 0, the limits $\lim_{t\to\infty} x_2^{(t)}$ and $\lim_{t\to\infty} x_4^{(t)}$ exist. This implies the existence of the limit $\lim_{t\to\infty} \mathbf{x}^{(t)}$ for any $\mathbf{x} \in S^2$. We can show for $\overline{V_{134}}, \overline{V_{124}}$, and $\overline{V_{123}}$ using the same argument. □

Next, we would like to describe the limit set of

$$\overline{V_{123}}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2), \\ x_2' = x_2(1 + x_2^r x_3 - x_1^{r+1}), \\ x_3' = x_3(1 - x_2^{r+1}), \end{cases} \qquad (6.5.2)$$

for any $\mathbf{x} \in int(S^2)$, the rest of the operators could be studied using similar technique. We denote $I_S^2 = \{(x_1, x_2) \in I^2 \mid x_1, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1\}$, and since $x_1 + x_2 + x_3 = 1$ it is sufficient to study the mapping $U_{123}: I_S^2 \to I_S^2$ defined by

$$U_{123}(\mathbf{x}) = \begin{cases} x_1' = x_1(1 + x_1^r x_2), \\ x_2' = x_2(1 + x_2^r(1 - x_1 - x_2) - x_1^{r+1}). \end{cases}$$

Note that for any $(x_1, x_2) \in I_S^2$, we have a corresponding $(x_1, x_2, 1 - x_1 - x_2) \in S^2$. By Proposition 6.7, we know that $\omega(\mathbf{x}) \subset \partial S^2$ for any $\mathbf{x} \in int(S^2)$. Since $x_1^{(t+1)} > x_1^{(t)}$ for any $t \in \mathbb{N}$, it is clear that $\mathbf{e}_2$ and $\mathbf{e}_3$ do not belong to $\omega(\mathbf{x})$. In fact, if we denote

$$H_1 = \{\mathbf{x} \in int(S^2) \mid x_2^r x_3 - x_1^{r+1} \geq 0\} \qquad (6.5.3)$$

and

$$H_2 = \{\mathbf{x} \in int(S^2) \mid x_2^r x_3 - x_1^{r+1} < 0\}, \qquad (6.5.4)$$

the trajectory of any initial point $\mathbf{x} \in H_1$ will eventually be in H_2 due to monotonicity of $x_1^{(t)}$ and $x_3^{(t)}$. Moreover, H_2 is invariant with respect to operator $\overline{V_{123}}$, since

$$x_2'^r x_3' - x_1'^{r+1} < x_2^r x_3 - x_1^{r+1} < 0.$$

Consequently, we have $\omega(\mathbf{x}) \subset \Gamma_{13} \setminus \{\mathbf{e}_3\}$ for any $\mathbf{x} \in int(\Gamma_{123})$.

Furthermore, despite $x_1^{(t)}$ being monotone increasing, we will show that $\mathbf{e}_1 \notin \omega(\mathbf{x})$. First, let

$$\Delta = \frac{x_2^r(1 - x_1 - x_2) - x_1^{r+1}}{x_1^{r+1}} \quad \text{and} \quad \Delta_{\mathbf{e}_1} = -\frac{x_2}{1 - x_1}$$

be the slope of lines connecting $\mathbf{x}$ with $\mathbf{x}'$, and $\mathbf{x}$ with $\mathbf{e}_1$ on the I_S^2 plane, respectively. For any $\mathbf{x} \in H_2$, we show that

$$\Delta' = \frac{x_2'^r(1 - x_1' - x_2') - x_1'^{r+1}}{x_1'^{r+1}} < \frac{x_2^r(1 - x_1 - x_2) - x_1^{r+1}}{x_1^{r+1}} = \Delta$$

for any corresponding $(x_1, x_2) \in I_S^2$.

In general, we have $\Delta^{(t+1)} < \Delta^{(t)}$, where

$$\Delta^{(t)} = \frac{x_2^{(t)r}\left(1 - x_1^{(t)} - x_2^{(t)}\right) - x_1^{(t)r+1}}{x_1^{(t)r+1}}.$$

is the slope connecting the point $\mathbf{x}^{(t)}$ with $\mathbf{x}^{(t+1)}$. Also, this implies $\Delta^{(t)} \to -1$ as $t \to \infty$. The following theorem estimates the limiting set of $\overline{V_{123}}$:

Theorem 6.17 Let H_1 and H_2 be defined as (6.5.3) and (6.5.4), respectively. If $\mathbf{x}^{(0)} \in H_1$, then there exists $N_0 \in \mathbb{N}$ such that $\mathbf{x}^{(t)} \in H_2$ for any $t > N_0$. Furthermore, if $\mathbf{x}^{(0)} \in H_2$, then $\omega\left(\mathbf{x}^{(0)}\right) \subset \left\{\mathbf{x} \in \Gamma_{13} \mid x_1 \in \left(x_1^{(0)} + x_2^{(0)}, x_1^{(0)} - \frac{x_2^{(0)}}{\Delta^{(0)}}\right)\right\}$, where $x_1^{(0)} - \frac{x_2^{(0)}}{\Delta^{(0)}} < 1$.

Proof. Recall that for any $\mathbf{x}^{(0)} \in H_1$, we have $\mathbf{x}^{(t)} \in H_2$ as $t \to \infty$. Now suppose $\lim_{t\to\infty} \mathbf{x}^{(t)} = \mathbf{e}_1$ for some $\mathbf{x} \in H_2$. Due to $\Delta^{(t+1)} < \Delta^{(t)}$, it is necessary that $\Delta > \Delta_{\mathbf{e}_1}$. However, this is not true since for any $x_1 > \frac{1}{2}$ we have

$$\Delta = \frac{x_2^r(1 - x_1 - x_2)}{x_1^{r+1}} - 1 < \frac{1 - x_1 - x_2}{1 - x_1} - 1 = -\frac{x_2}{1 - x_1} = \Delta_{\mathbf{e}_1}.$$

Furthermore, for any $\mathbf{x}^{(0)} \in H_2$ and $0 < t \in \mathbb{N}$, we have

$$-1 < \Delta^{(t)} < \Delta^{(0)} = \frac{x_2^{(0)r}\left(1 - x_1^{(0)} - x_2^{(0)}\right) - x_1^{(0)r+1}}{x_1^{(0)r+1}}.$$

Then, we construct two equations of line using the gradients -1 and $\Delta^{(0)}$, and obtain

$$x_2 = -x_1 + c_1,$$
$$x_2 = \Delta^{(0)} x_1 + c_2.$$

If both lines intercept the initial point $\left(x_1^{(0)}, x_2^{(0)}\right) \in I_S^2$, we have $c_1 = x_1^{(0)} + x_2^{(0)}$ and $c_2 = x_2^{(0)} - \Delta^{(0)} x_1^{(0)}$. Therefore, both lines intercept the x_1-axis at $x_1 = x_1^{(0)} + x_2^{(0)}$ and $x_1 = x_1^{(0)} - \frac{x_2^{(0)}}{\Delta^{(0)}}$. Here, $x_1^{(0)} - \frac{x_2^{(0)}}{\Delta^{(0)}} < x_1^{(0)} - \frac{x_2^{(0)}}{\Delta_{\mathbf{e}_1}} = x_1^{(0)} + \frac{x_2^{(0)}\left(1 - x_1^{(0)}\right)}{x_2^{(0)}} = 1$. Using these facts, one could show that $x_1^{(t)} \in \left(x_1^{(0)} + x_2^{(0)}, x_1^{(0)} - \frac{x_2^{(0)}}{\Delta^{(0)}}\right) \subset \left(x_1^{(0)} + x_2^{(0)}, 1\right)$ as $t \to \infty$. □

In summary, the trajectory of $V_{r,3}$ on the 2-dimensional faces of the 3-dimensional simplex converges to either $int(\Gamma_{13})$ or $int(\Gamma_{24})$ for any initial point taken from the face.

CHAPTER SEVEN

CONCLUSION

We introduced a general class of Lotka-Volterra operator defined on an $(m-1)$-dimensional simplex in Chapter 2. Such class of operator encompasses most of the non-ergodic Lotka-Volterra operators we recall within the same chapter. From the general operator, we derive and study the dynamics of several classes of Lotka-Volterra operators defined on the 2 and 3-dimensional simplexes.

First, we find that a bijective Lotka-Volterra operator is not necessarily **f**-monotone. Sufficient conditions are given in Chapter 3, under which a class Lotka-Volterra operator is a homeomorphism or bijection. This is contrary to the M-LV type Lotka-Volterra operator studied by Mukhamedov and Saburov (2017) which is **f**-monotone and also a bijection.

In Chapter 4, we generalise a problem on non-ergodicity of a class of Lotka-Volterra operators presented in Chapter 2. We established the uniqueness of its interior fixed point, and by constructing a Lyapunov function the omega-limiting set is estimated to lie in the boundary of the 2-dimensional simplex. We find that the path taken by a trajectory will visit each neighbourhood of a vertex periodically in a long run, and the time it spends in each neighbourhood increases for every succeeding visit. The sequence of such trajectory is constructed. Here, we find that our generalisation is also a non-ergodic operator.

We observe that the convex combination of two Lotka-Volterra operators in Chapter 5 is also Lotka-Volterra, shown to be a reduced case of the general operator introduced at the end of Chapter 2. Unlike previous study, where the two operators considered are of similar order, we find that, under some parameter less than one; fixed points exist in the interior of the edges of the 2-dimensional simplex when the order of the non-ergodic operator is higher than the regular one. Locally, those fixed points, if exist; are shown to be either repelling or saddle-node. Recall that we impose a condition to its parameter under which the edges fixed points are saddle-node. Under such condition, the interior fixed point is shown to be unique, and it is possible to construct a Lyapunov function. From the Lyapunov function, the omega limiting set of any initial point taken from the interior besides the centre is estimated to lie in the boundary of the simplex. Though, we later show that such set is a singleton, which member is one of the vertices of the simplex. We prove that, under given condition; its trajectory converges for any initial point taken from the simplex, hence the operator is regular. Another way to view this result is this: while it is known that our operator is non-ergodic when parameter equals one, here we show that it is regular for parameter close enough to one. This is contrary to the case of two operators of similar order considered by Jamilov and Reinfelds (2021) where it was shown that the resulting operator is non-ergodic for parameter close to one.

In Chapter 6, we consider a class of Lotka-Volterra operator defined on the 3-dimensional simplex. Despite the restriction on the operator, we consider it as an extension to the study we did on Chapter 4 in terms of its simplex. Interestingly, two of the edges of the simplex are found to consist of uncountable fixed points which are shown to be non-hyperbolic. We also prove the uniqueness of its interior fixed point and its repelling nature. Then, by constructing several Lyapunov functions we show that its omega-limiting set lies in the edges of the simplex. Moreover, for some initial point close enough to the edge, we find the path taken by its trajectory. We deduce the existence of a subset of the interior of the simplex from which, over the long run; the trajectory eventually follows the path. Here, the trajectory will visit each neighbourhood of the simplex periodically, and the time it spends in each neighbourhood increases every subsequent visit – similar to the operator in Chapter 4. Later, by constructing the sequence of such trajectory, we show that the Cesaro sum of the trajectory does not converge. As for the dynamics of the operator restricted to a 2-dimensional face of the simplex, we show that such operator is regular, and its limiting set lies in the boundary – specifically, on the edge where those uncountable fixed points exist.

7.1 FUTURE RESEARCH

In our thesis, we consider several restrictions on our operator due to its complexity. Nevertheless, limiting our study to lower dimensional simplex gives us an insight to the possibility of extending our problem to the general case.

In Ganikhodjaev (1993), it was proven that a Volterra QSO defined on a finite simplex is a homeomorphism. Hence, it is natural to consider similar problem for the Volterra CSO. In our case, despite extending the problem to a higher order Lotka-Volterra operator, we limit ourselves to the 2-dimensional simplex. Even for the cubic case, bijectivity of a Lotka-Volterra operator defined on a finite dimensional simplex is still an open problem.

Recall our general operator introduced in Chapter 2. Under some condition, as shown in Chapter 4 and 6; we find that it is non-ergodic for the one defined on the 2 and 3-dimensional simplexes. Here, we may deduce that it is non-ergodic for the one defined on a finite dimensional simplex, and future research pertaining to it seems plausible.

As for the convex combination of two LV operators in Chapter 5, our results are limited to a parameter which enables us to construct a Lyapunov function. Due to the complexity in constructing a Lyapunov function for the rest of the parameter, any research following that direction will be an interesting one.

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PUBLICATIONS

9.1 PUBLISHED PAPERS

Pah, C. H., & Rosli, A. (2022a). Bijectivity of a class of Lotka-Volterra operators defined on 2D-Simplex. In *Springer eBooks*, 319–327.

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Mukhamedov, F., Pah, C. H., & Rosli, A. (2024). On a convex combination of Lotka-Volterra operators. *Examples and Counterexamples, 5*, 100133.

9.2 SUBMITTED PAPERS

Mukhamedov, F., Hee, P. C., & Rosli, A. (2024). Ergodicity of replicator equation with historic behavior perturbation.

WORKSHOPS AND CONFERENCES

The author attended the following workshop and conference:

i. Guest speaker in "Non-associative Algebra, Dynamical Systems and Their Application in Biology" workshop organized by Fakulti Sains Komputer dan Matematik, UiTM Shah Alam campus on 28th of July 2022.

ii. Presenter in "International Conference on Mathematics: Computational and Theoretical Sciences (ICMCTS) 2023" organized by Department of Computational and Theoretical Sciences, IIUM Kuantan campus on 8th to 10th of August 2023.