

**λ -MODEL WITH POTTS COMPETING INTERACTIONS
ON CAYLEY TREE OF ORDER TWO: GROUND
STATES AND PHASE TRANSITIONS**

BY

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ABSTRACT

In this research, we consider the λ -model with and without competing interaction on Cayley tree of order two. Description of ground states becomes one of the main elements to study as phase diagram of Gibbs measure for a Hamiltonian is close to the phase diagram of isolated ground states of the Hamiltonian. For the λ -model on infinite Cayley tree, we describe the set of periodic and weakly periodic ground states corresponding normal subgroup of the Cayley tree group representation. We construct 81 different combination of configurations and classify the configurations under 10 different regions so that the configurations will achieve ground states. We describe periodic and weakly periodic ground states for the considered model by using periodic and weakly periodic configurations. For the second result of the research, we consider λ -model with competing Potts interaction on Cayley tree of order two. As explained in previous section, we describe the periodic ground states for the considered model. Note that for this model, we have 12 different regions for the configurations to achieve ground states. For some domain of interactions strength, the configuration of periodic ground states cannot be achieved. By using Kolmogorov criteria, Gibbs measures for this model was described by deriving infinite volume distribution using given finite-dimensional distributions and find the probability measures with given conditional probability. By considering translation invariant Gibbs measure, we analyse the system of equations derived and study the phase transition phenomenon by proving the existence of multiple translation-invariant solutions for the system of equations. Phase transitions occurs if there exist two or more solutions.

خلاصة البحث

في هذا البحث تمت دراسة لنموذج- λ مع التفاعل المتنافسة وبدونها على شجرة كايلى من الرتبة الثانية. ويصبح وصف الحالات الأرضية أحد من العناصر الرئيسية التي تجب دراستها حيث أن مخطط الطور لمقياس جيبس لهاميلتوني قريب من مخطط الطور للحالات الأرضية المعزولة لهاميلتوني. تم وصف مجموعة الحالات الأرضية الدورية والدورية الضعيفة المقابلة بالنسبة لنموذج- λ على شجرة كايلى اللاهائية للمجموعة الفرعية العادية لتمثيل مجموعة شجرة كايلى. وقام الباحث ببناء 81 مجموعة مختلفة من التكوينات وتصنيف التكوينات ضمن 10 مناطق مختلفة بحيث تحقق التكوينات حالات أرضية. وأوضح الحالات الأرضية الدورية والدورية الضعيفة للنموذج المدروس باستخدام تكوينات دورية ودورية ضعيفة. بالنسبة للنتيجة الثانية في هذا البحث، فتمت دراسة لنموذج- λ مع التفاعل بوتس المتنافسة على شجرة كايلى من الرتبة الثانية. وتم وصف الحالات الأساسية الدورية للنموذج المدروس كما وصف في القسم الأول. لاحظ الباحث أنه بالنسبة لهذا النموذج، لدينا 12 مناطق مختلفة للتكوينات لتحقيق حالات أرضية. وتكوين الحالات الأرضية الدورية لا يمكن أن يتحقق لبعض مجالات قوة التفاعلات. وتم وصف مقاييس جيبس لهذا النموذج من خلال اشتقاق توزيع الحجم اللاهائي باستخدام توزيعات ذات أبعاد محدودة وإيجاد مقاييس الاحتمالية مع الاحتمال الشرطي المحدد باستخدام معايير Kolmogorov. وقام الباحث بتحليل نظام المعادلات المشتقة ودراسة ظاهرة انتقال الطور عن طريق إثبات وجود حلول متعددة للترجمة الثابتة لنظام المعادلات من خلال النظر في مقاييس جيبس الثابت للترجمة. تحدث انتقالات الطور إذا كان هناك حلان أو أكثر.

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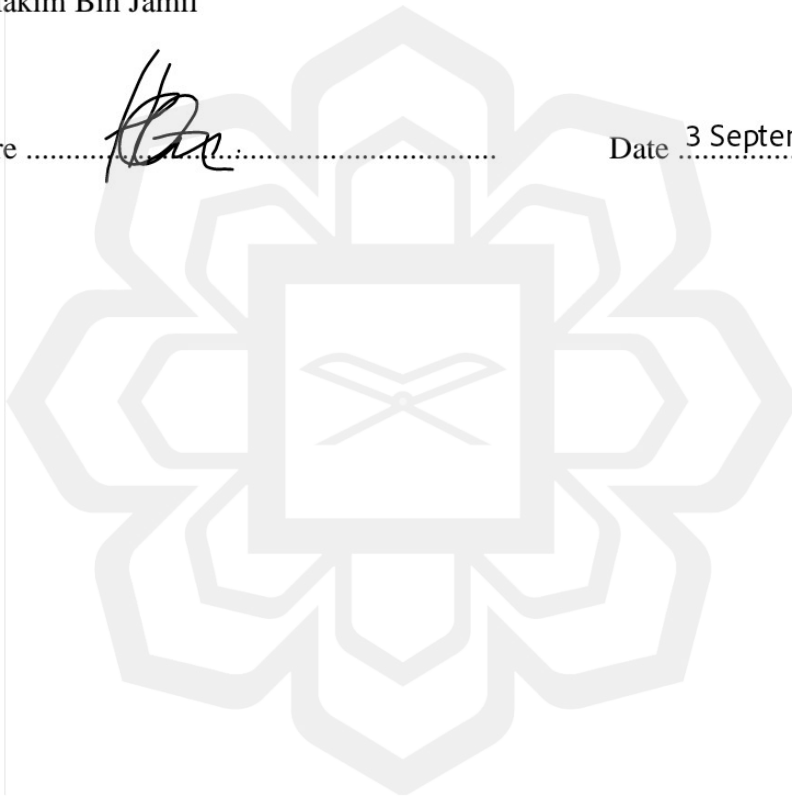
I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

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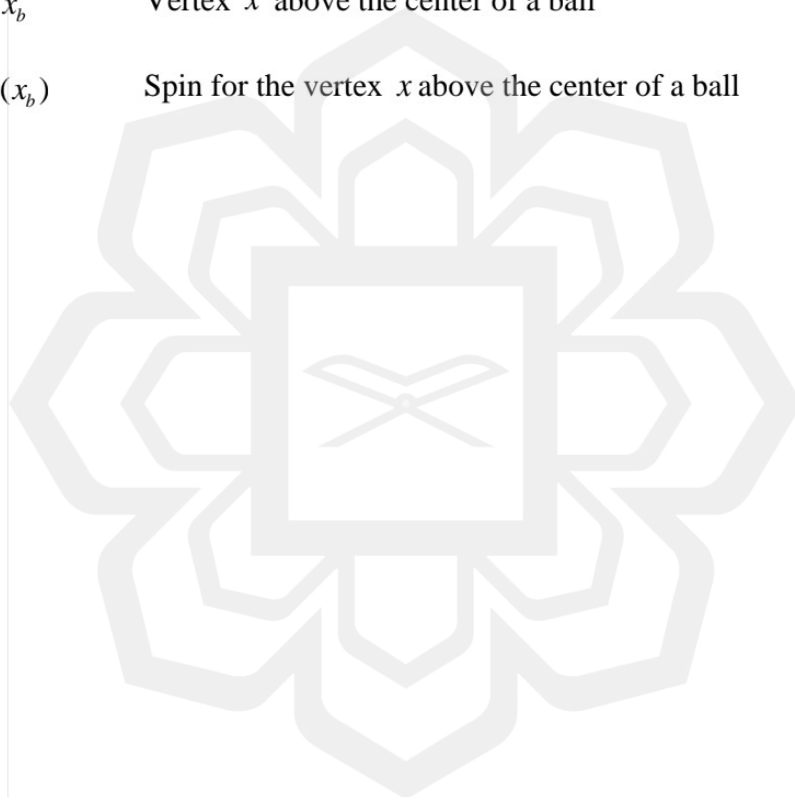


LIST OF SYMBOLS

Symbol

Λ	Finite volume
$d(x, y)$	Distance between x and y
V	Set of Vertices
$\sigma(x)$	Spin variable of x
$\Omega(\Lambda)$	Set of all configurations on Λ
T	Temperature
T^*	Critical Temperature
Γ^k	Cayley tree of order k
$\langle x, y \rangle$	Nearest-neighbor
$\succ x, y \prec$	Next Nearest-neighbor
$H(\sigma)$	Hamiltonian of configurations
W_n	Set of all vertices at n^{th} level
\mathbb{R}	Real number
λ	Lambda function
b	Ball
M	Set of balls
S_x	Successor of x
C_{b_x}	Center of ball
C_{b_\downarrow}	Vertex positioned below the center of a ball
x_\downarrow	Spin value below from center of a ball

φ_b	Configurations in a ball
$U(\sigma_b)$	Energy of configurations in a ball
$H_{\{a_i\}}$	Normal subgroup with consist of even number of a_i in text of elements in vertices x .
$\sigma(C_{b_\downarrow})$	Spin for the vertex positioned below the center of a ball
$\sigma(C_b)$	Spin for the center of a ball
x_b	Vertex x above the center of a ball
$\sigma(x_b)$	Spin for the vertex x above the center of a ball



CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION

In statistical mechanics, scholars are interested with the average properties of a mechanical system. Some examples of mechanical system are water in a bottle, the atmosphere inside a room and the number of atoms in a magnet. This kind of system are normally made up of a large number of components, usually molecules. However, in order to study the component of the system, the observer has limited capability to consider all of the component. As far as researchers are concerned, to consider an actual system with many (abstractly speaking, infinitely many) degrees of freedom, we cannot consider all properties of such a system into account without putting some exceptions. Consciously, researchers had no choice and neglected certain features. One of the way to overcome this restrictions, researchers start to specify a few average quantity of the system such as its density, pressure or temperature. Throughout this specifications, the main objective of statistical mechanics is by given only a knowledge of the microscopic interactions between the components, researchers intend to predict and verify the relation with the observable macroscopic properties of the components. This problem can be well-explained by the mathematical framework. For an example, microscopic systems behavior for the freezing and boiling process of water. This kind of process can be well-described through the simplified model of structure constructed from the interactions of between atoms contained in the object, by this example scenario is water itself. In order to study the problem related to the equilibrium statistical mechanics, the choice of the Hamiltonian for the systems of interacting particles are important (Baxter,

1982). On the other hand, Georgii (1988) stated that the main goal of equilibrium statistical mechanics is to describe all limit Gibbs distributions if given a Hamiltonian of the system. This problem can be completely solved only in some relatively simple cases. In the given certain condition, if there are only binary interactions in the system considered, then the problem of describing the limit Gibbs distributions is significantly simplified. Phase diagram of isolated (stable) ground states of the Hamiltonian is close to the phase diagram of Gibbs measures for a Hamiltonian. A periodic ground state corresponds to a periodic Gibbs measure (Sinai (1982); Minlos (2000)) at low temperatures. There are many mathematical models that have been constructed by the researchers in order to understand this system, e.g. Ising model, Potts Model, and many more.

1.2 LITERATURE REVIEW

The main objective of statistical mechanics studies is to describe the relation between observable macroscopic properties of the components given by only knowledge of microscopic interactions between components. As known by researchers, Ising model is the simplest model in statistical mechanics which has wide practical applications and theoretical interest (Georgii (1988); U. A. Rozikov (2013)). There are several papers which are related to the study for description of this set for the Ising model on a Cayley tree. The exact solution of an Ising model with competing interactions on a Cayley tree was studied by Ganikhodjaev et al. (2003). However, this Ising model has lacking in completing the result about all Gibbs measures. Early in 1982, Vannimenus introduced a method on describing phase diagram for Ising model with competing interactions. This phase diagram constructed from numerical value of system of equation. For this problem, Ising model with next nearest-neighbor interactions on the Cayley tree was

considered. This method of describing the phase diagram lead to the increasing of interest for the researchers. Ganikhodjaev et al. (2011) described the phase diagram of an Ising system with competing binary, prolonged ternary and next-nearest interactions on a Cayley tree. Ground states related were also been studied for this ising model. Ising model on a general Cayley tree with competing interactions of next-nearest-neighbour of two types; prolonged and one-level K -tuple interactions was studied (Ganikhodjaev & Zakaria, 2011). Ganikhodjaev & Rodzhan (2014) studied the phase diagram for the Ising model that define on Cayley tree-like lattice; pentagonal chandelier, with competing one-level pentagon interactions. In this research works, pentagonal chandelier can be viewed as another geometrical representation on Cayley tree of order 5. The phase diagram was investigated and it showed the appearance of several features and modulated phase arising from frustration effects introduced by one-level pentagon interaction for several ranges of the competing parameters. Later on, new weakly periodic (non-periodic) Gibbs measures corresponding to normal subgroups of indices two in the group representation of the Cayley tree of order five and six was studied by Ganikhodjaev et al. (2017).

On the Other hand, Potts (1952) introduced Potts model as generalization of the Ising model. This Potts model encompasses a number of problems in statistical physics (see also (Wu, 1982)) due to its spin value can be consider more than two components. By these facts, this Potts model grows interest among the researchers. The critical behavior of the two-dimensional, q -state Potts model, using finite-size scalling and transfer matrix methods was investigated (Nightingale & Schick, 1952). This model simply described the special class of statistical mechanics system. Later on, the Potts model with competing interacions on the Cayley was investigated. This model is more complex and has rich structure of ground states. Ganikhodjaev (1990) described the

pure phase of the ferromagnet Potts with three states of Bethe lattice of order two. Later on, Botirov & Rozikov (2007) considered Potts model with three spin values and with competing interaction of radius $k = 2$. In this paper, they described the ground states and proved non uniqueness of Gibbs measures for the model by using contour method. Ganikhodjaev et al. (2007) studied the Potts model on a Cayley tree in the presence of competing two binary interactions and magnetic field. This paper solved for the phase transitions problem and the critical curve for the phenomenon of phase transitions was found. As the continuation of the research by Ganikhodjaev (1990), phase diagram of the three-states Potts model with competing nearest neighbor and next nearest-neighbor interactions on a Cayley tree has been numerically obtained (Ganikhodjaev et al., 2008). The two-dimensional q -component Potts model is equivalent to a staggered ice-type model. It is deduced that the model has first-order phase transitions for $q > 4$, and a higher-order transition for $q \leq 4$. The free energy and latent heat at the transition were calculated (Baxter, 2011). Periodic Gibbs measure was studied for the Potts model on the Cayley tree (Rozikov & Khakimov, 2013). In this research, the condition for the considered model with nonzero external fields admits periodic Gibbs measure was found.

It is natural to consider more complicated models than the Potts one, so called λ -model. This λ -model was first introduced by Rozikov (1998). In this paper, limit Gibbs measures for the λ -model on the Bethe lattice was described by considering spin values $\{-1, +1\}$. Motivated by this research potential, Mukhamedov (2004) continued the study by investigating the factor associated with the unordered phase of λ -model on a Cayley tree. Phase transition phenomenon are not considered in this research work. On the latest research works, Mukhamedov et al. (2017) described the ground states for the

λ -model on Cayley tree of order two. As the continuation of the studies, phase transition phenomenon was studied for the considered model (Mukhamedov et al., 2018). As far as concern, there are not so many research paper related to the λ -model on Cayley tree.

The phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of the Hamiltonian. A periodic ground states corresponds to a periodic Gibbs measure (Sinai (1982); Minlos (2000)) at low temperatures. This is the main motivation why researchers intend to described the Gibbs measure of lattice models. Weakly periodic ground states of Ising model with competing interactions on cayley tree was described (Rozikov & Rahmatullaev, 2008). For the past few years, ground states related problem also were studies for this Potts model. Rahmatullaev (2013) described weakly periodic Gibbs measures and ground states for the Potts model with competing interactions on the Cayley tree. Some explicit formula of the free energies and entropies (according to vector-valued boundary conditions (BCs) are obtained for the Potts model on the Cayley tree which are corresponding to weakly periodic Gibbs measures. Later on, periodic and weakly periodic ground states for the Potts model with competing interactions on the Cayley tree was studied (Rahmatullaev, 2016).

1.3 RESEARCH OBJECTIVES

As mentioned earlier, the description of ground states become one of important feature as a ground states corresponds to a Gibbs measures at low temperatures. For this thesis, we considered λ –model on Cayley tree of order two. There are several objectives were set up for this research. For the first part of the research, we consider λ – model on the Cayley tree of order two. For this considered model on Cayley tree of order 2 , we intend

- i. to describe the periodic ground states for the λ -model on Cayley tree;.

- ii. to describe the weakly periodic ground states.

For the second part of this project, we consider λ -model with Potts competing interaction on Cayley tree of order two. For the λ -model with Potts competing interactions on Cayley tree, we intend

- iii. to describe the periodic ground states;
- iv. to construct the Gibbs measures for the λ -model;
- v. to analyze the system of equations for the considered model and prove the existence of phase transition phenomenon.

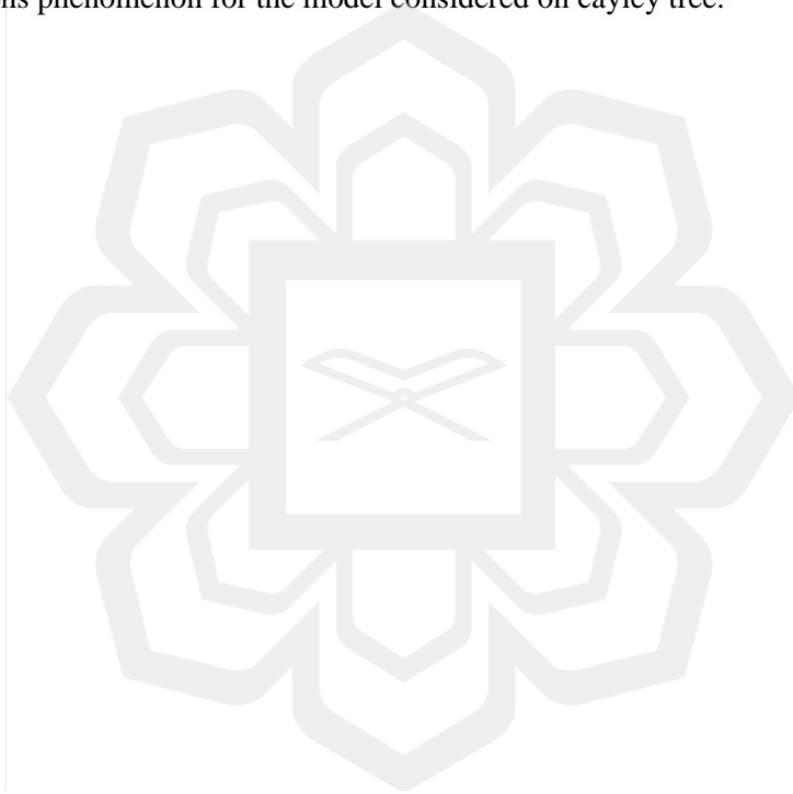
1.4 OVERVIEW OF THE THESIS

This thesis consist of three parts.

In chapter 2, we will discuss about the basic notion and setting that had been used to complete this research. We provide the definition of the Cayley tree, Gibbs measures, ground states, the model considered for this research and phase transition phenomenon. As mentioned, there are two type of models we considered for this research. In order to complete the first and second parts of the objectives, we consider λ -model on Cayley tree of order two. As for the third, fourth and fifth objectives of this research, we consider λ -model with Potts competing interactions on Cayley tree of order two.

In chapter 3, we focus on completing the first and second objectives of my research. Note that this research are coninuation from my previous studies during my master study. In this chapter, λ -model on Cayley tree was introduced. Then, the periodic and weakly periodic ground states for the considered model are described. All proof of the existence of this ground states, either periodic or weakly periodic cases wre provided in this chapter.

In chapter 4, we considered the λ -model with competing Potts interactions on Cayley tree of order two. Noted that the model has additional Potts interaction compare to previous model in chapter 3. In this chapter, we will cover the next three objectives for my research. For the first part of this chapter, periodic ground states were considered for the model. For the second part, we will construct the Gibbs measures for the considered model. For this part, we proved for the consistency and efficiency of the Gibbs measures constructed. For the third part, we proved the existence of phase transitions phenomenon for the model considered on Cayley tree.



CHAPTER TWO

PRELIMINARIES

2.1 CAYLEY TREE

In this chapter, we will introduce the notion of Cayley tree, lambda model, Gibbs measure, ground state and phase transition. We also discuss the group structure of Cayley tree that associate with the vertex of the Cayley tree.

Let $\Gamma^k = (V, L)$ be a Cayley tree of order k , i.e, an infinite tree such that exactly $k + 1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k . For an arbitrary vertex $x_0 \in V$, we put

$$W_n = \{x \in V \mid d(x_0, x) = n\}, V_n = \bigcup_{m=0}^n W_m, L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

where $d(x, y)$ is the distance between x and y in the Cayley tree, i.e., the number of edges of the path between x and y .

Let G_k denote the free product of $k + 1$ cyclic groups $\{e, a_i\}$ of order 2 with generators $a_1, a_2, \dots, a_{k+1}, a_i^2 = e$.

There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order k and the group G_k (Rozikov & Khakimov, 2013).

For the sake of completeness, we show how to construct this correspondence. We choose an arbitrary vertex $x_0 \in V$ and associate it with the identity element e of the group G_k . Since we may assume that the graph under consideration is planar, we associate each neighbor of x_0

(i.e., e) with a single generator $a_i, i = 1, 2, \dots, k + 1$, where the order corresponds to the positive direction, see Figure 2.2.

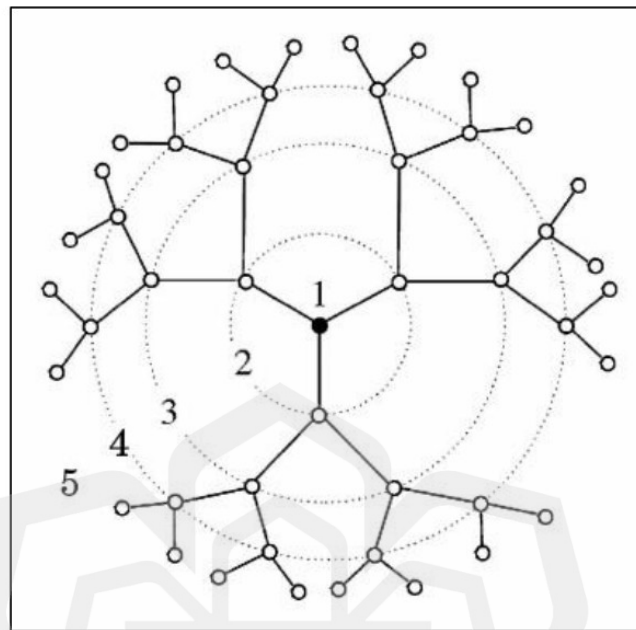


Figure 2.1: The Cayley tree of order 2, Γ^2

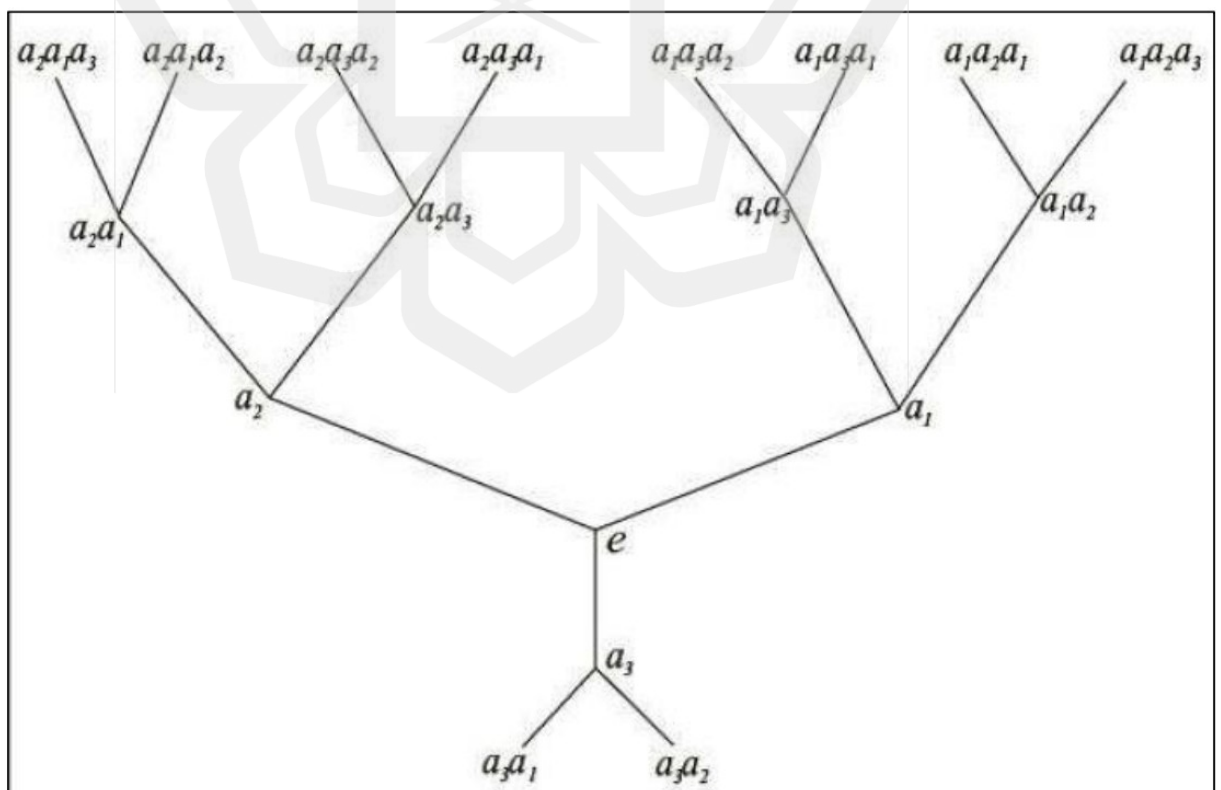


Figure 2.2: The Cayley tree Γ^2 and Elements of the Groups Representation of vertices

For every neighbor of a_i , we introduce words of the form $a_i a_j$. Since one of the neighbors of a_i is e , we put $a_i a_j = e$. The remaining neighbors of a_i are labeled according to the above order. For every neighbor of $a_i a_j$, we introduce words of length 3 in a similar way. Since one of the neighbors of $a_i a_j$ is a_i , we put $a_i a_j a_l = a_i a_l$. The remaining neighbors of $a_i a_j$ are labeled by words of the form $a_i a_j a_l$, where $i, j, l = 1, 2, \dots, k+1$, according to the above procedure. This agrees with the previous stage because $a_i a_j a_j = a_i a_j^2 = a_i$. Continuing this process, we obtain a one-to-one correspondence between the vertex set of the Cayley tree Γ^k and the group G_k .

The representation constructed above is said to be right because, for all adjacent vertices x and y and the corresponding elements $g, h \in G_k$, we have either $g = ha_i$ or $h = ga_j$ for suitable i and j . The definition of the *left* representation is similar.

For the group G_k (or the corresponding Cayley tree), we consider the left (right) shifts. For $g \in G_k$, we put

$$T_g(h) = gh(T_g(h) = hg)$$

for all $g \in G_k$. The group of all left (right) shifts on G_k is isomorphic to the group G_k .

Each transformation S on the group G_k induces a transformation S on the vertex set V of the Cayley tree Γ^k (U. A. Rozikov, 2013). From now on, we identify V with G_k .

Theorem 2.1.1 [U. A. Rozikov (2013)]: *The group of left (right) shifts on the right (left) representation of the Cayley tree is the group of translations.*

By the group of translations we mean the automorphism group of the Cayley tree regarded as a graph. Recall that a mapping ψ on the vertex set of a graph G is

called an automorphism of G if ψ preserves the adjacency relation, i.e., the images $\psi(u)$ and $\psi(v)$ of vertices u and v are adjacent if and only if u and v are adjacent.

For each $x \in G_k$, let $S(x)$ denote the set of immediate successor of x , i.e., if $x \in W_n$, then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x , i.e.,

$S_1(x) = \{y \in G_k : \langle x, y \rangle \in L\}$. The set $S_1(x) \setminus S(x)$ is a singleton. Let x_\downarrow denote the (unique) element of this set.

Spin values are introduced as the value at the vertex on a Cayley tree. Assume that spin takes its values in the set $\Phi = \{1, 2, \dots, q\}$. By a configuration on V denoted by σ we mean a function taking $\sigma : x \in V \rightarrow \sigma(x) \in \Phi$. The set of all configurations coincides with the set $\Omega = \Phi^V$.

2.2 λ -MODEL

Lattice model is a model defined on a lattice, e.g. square lattice or Cayley tree.

Researchers can study the phase transition phenomenon by describing all limits Gibbs measures corresponding to the Hamiltonian defined on the lattice model. Ising model was first introduced by Ising (1925) that consider spin values $\{-1, +1\}$. For this Ising model, the interaction between two vertices are introduced as multiplication of the spin value of the vertices. This model was considered as the simplest model that attract wide theoretical interest. Later on, Potts model was introduced as generalization of the Ising model, has more than two components. λ -model was