A P-FINITE ELEMENT METHOD OF A THREE DIMENSIONAL NON-UNIFORM ASYMMETRIC BEAM STRUCTURE OF ARBITRARY POLYNOMIAL FUNCTIONS

BY

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ABSTRACT

Tapered beams are commonly used in civil, aerospace or mechanical engineering structures as they can reduce its structural weight without sacrificing the strength and flexibility. Tapered beams are also used to satisfy aesthetic or architectural requirement. Most of numerical methods to analyze tapered beam structures are using a Galerkin's finite element approach where the beam is divided into a number of elements to obtain accurate result. The beam stiffness matrix is usually obtained through integration of each element by assuming a shape function for the beam transversal deformations. Since the number of elements are big, therefore such an approach may affect computational time. In the present research, a different approach is conducted where an analytical formulation of a finite element stiffness matrix for a tapered, asymmetric beam element is developed by using a flexibility approach. The beam stiffness matrix is first divided into bending, axial and torsional matrices. For the bending stiffness matrix, to simplify the formulation and therefore to accelerate the numerical calculation, it is necessary to divide further the bending stiffness matrix into four sub-matrices. Each of the sub-matrices is a 4-by-4 matrix representing the bending stiffness matrix in three dimensional coordinate system. The key to the present approach lays on the formulation of the first sub-matrix, whereas the other three sub-matrices can be obtained from the first sub matrix by using direct, simple matrix operations. The first sub-matrix is constructed based on the flexibility approach where a two-steps analytical integration of second order, partial differential equations is performed. The partial differential equations are derived based on the Euler-Bernoulli governing equations for the three-dimensional bending deformations, where the transversal deformations of the beam are coupled due to the properties of the asymmetric cross section. After rearranging the transversal deformations in matrix forms, the resulting explicit forms of the differential equations contain rational functions with multi-polynomial functions on both numerator and denominator of the rational function. It is found that, in order to ensure the robustness of the integrations, the denominator functions should be expressed as the multiplication factor of their roots. By properly considering the boundary conditions of the beam under various load conditions, the results of the analytical integration are a 4-by-4 flexibility matrix. The final form of the first sub-matrix is the stiffness matrix which can be obtained by matrix inversion of the flexibility matrix. For the axial and torsional stiffness matrices, a similar approach is conducted but it is much simpler since it involves only first order differential equations. It is found that the present stiffness matrix contains logarithmic terms which are not occurred if one use direct Galerkin's finite element approach. The present finite element method can be considered as an analytical stiffness matrix formulation since no assumed shape functions used for the whole process of the formulation. Therefore, if the tapered functions of the beam geometry is given, only one element is sufficient to accurately simulate the beam deformation. To validate the present finite element method, a number of structural tapered beam having symmetric and asymmetric cross section are used and the results are compared with available analytical result or other software's such as Nastran. The results show that the present method gives the accuracy of more than 7 significant digits compared with the analytical solution. In all cases, the present method by using one element gives the result similar to Nastran convergent result where, in order to achieve

the convergence, a number of elements in Nastran are needed. It is expected that the finding of the present method can contribute further the development of finite element numerical simulation.

خلاصة البحث

تستخدم الدعامات المستدقة بشكل شائع في هياكل الهندسة المدنية والفضائية والميكانيكية لأنه يمكن تقليل وزنها الهيكلي دون التضحية بقوتها ومرونتها. كما تستخدم الدعامات المستدقة أيضاً لتلبية المتطلبات الجمالية أو المعمارية. إنّ معظم الأساليب العددية المستخدمة لتحليل هياكل الدعامة المستدقة تستخدم طريقة العناصر المحدودة لجاليركن، حيث يتم تقسيم الدعامة إلى عدد من العناصر للحصول على نتيجة دقيقة. وعادة ما يتم الحصول على مصفوفة صلابة الانحناء للدعامة من خلال تكامل كل عنصر من خلال افتراض دالة شكل للتشوهات المستعرضة للدعامة، ونظراً لأن عدد العناصر كبير، فقد تؤثر هذه الطريقة على الوقت الحسابي. يعرض هذا البحث طريقة مختلفة تستخدم الصياغة التحليلية لمصفوفة صلابة العنصر المحدود للدعامة المستدقة غير المتناظرة باستخدام نهج المرونة، حيث يتم أولاً تقسيم مصفوفة صلابة الانحناء إلى ثلاث مصفوفات: الانحناء، والمحورية، والالتواء. أمّا بالنسبة لمصفوفة صلابة الانحناء، فلتبسيط الصيغة، وبالتالي تسريع الحساب العددي، فإنه من الضروري تقسيم مصفوفة صلابة الانحناء إلى أربع مصفوفات فرعية، كل من هذه المصفوفات الفرعية عبارة عن مصفوفة ٤×٤ تمثل مصفوفة صلابة الانحناء في نظام إحداثيات ثلاثي الأبعاد. ويكمن مفتاح الطريقة المقترحة في صياغة المصفوفة الفرعية الأولى، في حين يمكن الحصول على المصفوفات الفرعية الثلاثة الأخرى من المصفوفة الفرعية الأولى باستخدام عمليات مصفوفية مباشرة وبسيطة. حيث يتم إنشاء المصفوفة الفرعية الأولى بناءً على مفهوم المرونة، حيث يتم إجراء تكامل تحليلي ذي خطوتين من الدرجة الثانية، ويتم تنفيذ المعادلات التفاضلية الجزئية. يتم اشتقاق المعادلات التفاضلية الجزئية استنادا إلى معادلات أويلر-بيرنولي التي تحكم تشوهات الانحناء ثلاثية الأبعاد، حيث تقترن التشوهات العرضية للدعامة بسبب خصائص المقطع العرضي غير المتماثل. وبعد إعادة ترتيب التشوهات العرضية في أشكال المصفوفة، تحتوي الأشكال الصريحة الناتجة من المعادلات التفاضلية دالّات نسبية مع دالّات متعددة الحدود في كل من البسط والمقام للدالة النسبية، وقد وجدت الدراسة أنه من أجل ضمان متانة عمليات التكامل، يجب التعبير عن دالّات المقام كعامل الضرب لجذورها. ومن خلال الأخذ بالاعتبار الشروط الحدية للدعامة تحت ظروف الحمل المختلفة، فإن نتائج التكامل التحليلي هي مصفوفة مرونة ٤×٤. والشكل النهائي للمصفوفة الفرعية الأولى هو مصفوفة الصلابة التي يمكن الحصول عليها عن طريق معكوس مصفوفة المرونة. أما بالنسبة لمصفوفات الصلابة المحورية والالتوائية، فيتم إجراء نهج مماثل ولكنه أبسط بكثير لأنه يتضمن فقط معادلات تفاضلية من الدرجة الأولى. كما وجدت الدراسة أن مصفوفة الصلابة الحالية تحتوي على حدود لوغاريتمية لا تنتج إذا استخدم المرء طريقة العناصر المحدودة لجاليركن. ويمكن اعتبار طريقة العناصر المحدودة الحالية بمثابة صيغة مصفوفة صلابة تحليلية حيث لا توجد دالّات شكل مفترضة تستخدم في العملية الكاملة للصياغة. لذلك، إذا كانت الوظائف المستدقة لهندسة الدعامة معلومة، فيكفي عنصر واحد فقط لمحاكاة تشوه الدعامة بدقة. وللتحقق من طريقة العناصر المحدودة الحالية، يتم استخدام عدد من الدعامات الهيكلية المستدقة ذات المقطع العرضي المتماثل وغير المتماثل، وتتم مقارنة النتائج بالنتائج التحليلية المتاحة أو البرامج الأخرى مثل (Nastran). وقد أظهرت النتائج أن الطريقة الحالية تعطي دقة أكثر من ٧ خانات معنوية مقارنة مع الحل التحليلي. وفي جميع الحالات، فإن الطريقة الحالية باستخدام عنصر واحد تعطي نتيجة مشابهة لنتيجة (Nastran) المتقاربة، حيث أن هناك حاجة إلى عدد من العناصر في (Nastran) من أجل تحقيق هذا التقارب. ومن المتوقع أن يساهم اكتشاف الطريقة الحالية في تطوير المحاكاة العددية للعناصر المحدودة.

APPROVAL PAGE

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DECLARATION

I hereby declare that this thesis is the result of my own investigations, except where otherwise stated. I also declare that it has not been previously or concurrently submitted as a whole for any other degrees at IIUM or other institutions.

S.M. Afzal Hoq

Signature ……………………………… Date ……………………

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……..……………………. ……………………
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This thesis is dedicated to

my beloved wife and daughter

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LIST OF SYMBOLS

A	Cross section area of a beam element
E	Modulus of elasticity
F_i , $i = A, B$	External Force at the tip or root
f_{ij}	Flexibility matrix coefficient
F_{Ay}	Concentrated force at the tip of the beam in y direction
F_{Az}	Concentrated force at the tip of the beam in z direction
F_{By}	Concentrated force at the root of the beam in y direction
F_{Bz}	Concentrated force at the root of the beam in y direction
G	Shear modulus
I	Moment of inertia
J	Polar moment of inertia
K_{ij}	Stiffness matrix
L	Length of element
L_{xy}	Arbitrary polynomial along the beam span
M	bending moment
M_0	Moment
M_i , $i = A, B$	Moment at the tip or root
M_{Ay}	Moment at the tip of the beam in y direction
M_{Az}	Moment at the tip of the beam in z direction
M_{By}	Moment at the root of the beam in y direction
M_{Bz}	Moment at the root of the beam in z direction
N	Internal force

P_0	Concentrated force
P_A	Load vector at point A
P_B	Load vector at point B
Q	Shear force
S_{ij}	Stiffness matrix coefficient
T	Torque
u_A	Deformation vector at point A
u_B	Deformation vector at point B
δ	Beam lateral deformation
θ	Beam lateral rotation
ϕ	Twist gradient
δ_{Ay}	Deflection at the tip of the beam in y direction
θ_{Az}	Rotation at the tip of the beam in z direction
δ_{By}	Deflection at the root of the beam in y direction
θ_{Bz}	Rotation at root of the beam in z direction

LIST OF ABBREVIATION

AFG	Axially Functionally Graded
BVP	Boundary Value Problem
CFM	Complementary Function Method
CAD	Computer-Aided Design
CB	Continuum Based
DOF	Degree Of Freedom
DTEM	Differential Transform Element Method
DTM	Differential Transform Method
FEM	Finite Element Method
FGM	Functionally Graded Material
PDE	Partial Differential Equation
GFEM	Galerkin Finite Element Method
EHP	Extended Hamilton's Principle
IGA	Isogeometric Analysis
MWR	Method Of Weighed Residuals
NURBS	Non-Uniform Rational Basis Spline
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
TGFEM	Trilinear Galerkin Finite Element Method
VAM	Variational-Asymptotic Method
VABS	Variational-Asymptotic Beam Sectional Analysis
MDPR	Minimum Denominator Rational Function
P-FEM	*P*-Finite Element Method
h-FEM	*h*-Finite Element Method
hp-FEM	*hp*-Finite Element Method

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

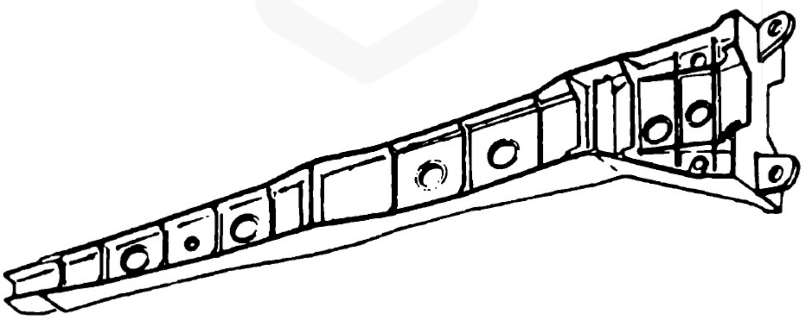
Finite element method (FEM) is a numerical procedure for solving differential equations occurring in a variety of problems in engineering such as structural analysis, thermodynamic, fluid dynamic, and electromagnetic as well as in medical science and in mathematical physics. FEM is well accepted due to its capability to treat complex geometry and irregular shape and boundary conditions by discretization of the model domain into a number of finite elements. The accuracy of the finite element approximation can be improved by increasing the number of elements, which is called h-FEM method, or by increasing the polynomial order of the finite element model, which is called p-FEM method, or by combining both methods, which is called hp-FEM method.

Tapered beams known as non-uniform beams or non-prismatic beams as shown in Figure 1.1 are frequently used in many civil engineering, mechanical engineering and aerospace engineering fields. The bridge girder structure shown in Figure 1.1(a) is designed by considering not only its structural strength requirement but also its architectural aspect. The tapered profile or shape shown in Figure 1.1(b) provides a maximum stiffness-to-mass ratio for earthquakes or other vibrations of the earth and wind load strength. The piston complicated geometry shown in Figure 1.1(c) is the result of extensive optimization analysis in order to produce the optimum design. In aircraft wing, the front and rear spar are commonly tapered beam where the profile height is bigger at the wing root and smaller at wing tip as shown in Figure 1.1.(d). In addition to less weight, since less material is required to manufacture tapered beams, this type of framing is more cost-effective than using all straight members.

(a) Bridge girder beam (taken from Zevaloss, 2016)

(b) Beam and column of frame structure
(taken from quora.com, 2020)

(c) Piston rod
(taken from McCune 2001)



(d) Integrally machined spar of aircraft wing (taken from Niu, 1988)

Figure 1.1 Examples of non-prismatic beam structures

To analyze the tapered beam structure, most of researches are using finite element methods based on stiffness formulation. The stiffness formulation is usually derived based on Galerkin's approach where a cubic polynomial shape function is assumed for the element's deformation. For a non-uniform structure as shown in Figure 1.2, the integration of the shape function will give polynomial function terms only. As it will be shown in Chapter 4, for the type of h-FEM approach, in order to obtain accurate result, one needs to do some convergence study to ensure that the accuracy is within acceptable requirement. Therefore, application of the h-FEM approach to a non-uniform beam model needs a well-practiced user in order to give accurate result. A non-competent user who used only single element or inappropriate number of elements to model the non-uniform beam structure may give significant error on the beam deformation, strain and stress.

Motivated to circumvent this problem, the present work is conducted to develop a p-finite element method for asymmetric, non-uniform beam such as shown in Figure 1.2. An in-house code is developed in MATLAB and the result is validated by comparing with analytical results and commercial software's such as Nastran.

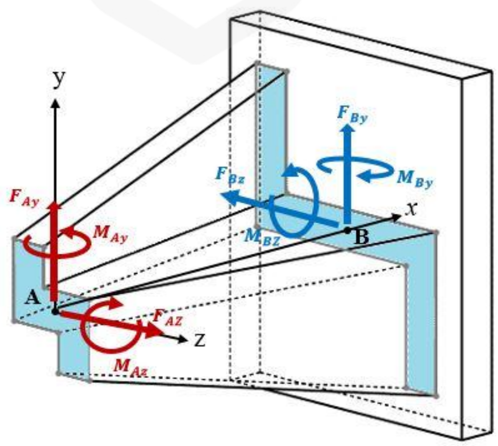


Figure 1.2 Non-uniform, asymmetric beam model

1.2 PROBLEM STATEMENT

Most of the formulations to construct stiffness matrix in the finite element method is based on the so-called matrix stiffness approach. The element in the stiffness matrix in this method is obtained by direct integration of the shape function. The shape function for the beam uniform element is a cubic polynomial function. However, the same cubic function is used also for beam of non-uniform element solved using the h-FEM method. This similar treatment of shape function for both uniform and non-uniform beams may attribute to the slow convergence of the h-FEM for the non-uniform beam model. In other word, if the number of elements to model the non-uniform beam is not sufficient, the accuracy of the result using h-FEM is low.

Therefore, the main research question can be stated as: is there any method that achieve a high level of accuracy for calculating the stiffness matrix of non-uniform beam without the need to increase the number of elements?

In the present research work, an attempt is conducted to answer the research question above by calculating the stiffness matrix indirectly, i.e. the first step is to calculate the flexibility matrix by using the flexibility approach. Since the shape function is not used, the dependence to the number of elements can be reduced. The second step is to obtain the stiffness matrix by performing matrix inversion of the flexibility matrix.

The second problem statement is related to the computational time needed to perform the invers matrix. This computational time can be reduced in the present work by dividing the stiffness matrix into four blocks that each has similar size of the matrix. The flexibility matrix is performed only to the first block where its matrix size is only ¼ of the element stiffness matrix. The other three blocks are obtained by simple matrix operation.

The third problem statement is related to the advantage of the matrix stiffness method. Compare to the flexibility approach, the matrix stiffness approach is well-accepted due the easiness to assemble the stiffness matrix and to impose the boundary conditions. The conventional flexibility approach has a complicated way to address the load and boundary conditions. In the present work, a non-conventional way of the flexibility approach is performed, i.e the flexibility approach is performed only to ¼ of the flexibility matrix where a certain statically determined beam is selected in order to form easily its flexibility matrix coefficients. Since it is inverted directly to the stiffness matrix, basically the final result is in the form of stiffness matrix. Therefore, the advantage of the matrix stiffness method is still maintained.

The forth problem statement is related to no treatment to asymmetric, non-uniform beam in the literature. The literature available only for either non-uniform beam or asymmetric beam but not for both asymmetric, non-uniform beam. The present work attempts to address the stiffness matrix of the asymmetric, non-uniform beam by using the same frame work of procedure.

1.3 RESEARCH PHILOSOPHY

The philosophy of the present research is to develop a p-FEM method to reduce the dependency on the number of elements required to accurately construct the stiffness matrix of asymmetric, non uniform beam by using a flexibility method. To reduce the dependency on the number of elements, the shape function assumption is not used, instead a direct, analytical integration is performed. To reduce the computational time, only a quarter of the flexibility matrix is formulated and the result is inverted directly to obtain the stiffness matrix. To retain the advantage of the matrix stiffness approach,